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ESTIMATES FOR WEAK-TYPE OPERATORS

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1. **Introduction.** When f is an integrable function on the interval $[0, 1]$, we denote by f^* its nonincreasing rearrangement and by f^{**} the average $f^{**}(t) = (1/t)\int_0^t f^*(s)\,ds$. The Lorentz space L^{pq}, $1 \leqq p \leqq \infty$, $1 \leqq q \leqq \infty$ consists of all functions f for which the norm

$$\|f\| = \left\{\int_0^1 [t^{1/p}f^{**}(t)]^q \frac{dt}{t}\right\}^{1/q}$$

is finite; the Lorentz space $L^{(pq)}$ is defined in the same way except that f^{**} is replaced by f^*. When $1 < p \leqq \infty$, L^{pq} and $L^{(pq)}$ coincide, up to equivalence of (quasi) norms (cf. [3], [4]). The spaces L^{pq}, $1 < p < \infty$, are the intermediate spaces $(L^1, L^\infty)_{\theta,q;K}$, $\theta = 1 - 1/p$, generated by the K-method of J. Peetre (cf. [2], [5], [7]). Note that $L^{(1\infty)}$ is the space usually referred to as "weak-L^1" and that the Orlicz space $L \log^+ L$ of functions f for which $|f| \log^+ |f|$ is integrable, is (cf. [1]) none other than the Lorentz space L^{11}. Thus

$$L \log^+ L = L^{11} \subseteq L^{1\infty} = L^1 = L^{(11)} \subseteq L^{(1\infty)} = \text{weak-}L^1.$$

From characterizations of the intermediate spaces $(L \log^+ L, L^1)_{\theta,q;K}$ and $(L^1, \text{weak-}L^1)_{\theta,q;K}$ obtained by the author in [1] and subsequently, there follow some new estimates for weak-type operators. In particular, we obtain a sharper form of a theorem of O'Neil [6] concerning operators that are simultaneously of weak-types $(1, 1)$ and (p, p), $1 < p \leqq \infty$.

2. **Intermediate spaces between $L \log^+ L$ and L^1.** The space of functions f for which the norm

$$\|f\| = \left\{\int_0^1 \left[t\left(\log\frac{1}{t}\right)^{\theta - 1/q} f^{**}(t)\right]^q \frac{dt}{t}\right\}^{1/q}$$

is finite, will be denoted by $A^{\theta q}$, $0 < \theta < 1$, $1 \leqq q \leqq \infty$. The corresponding space with f^{**} replaced by f^* in the previous definition, is denoted by $A^{(\theta q)}$. The following results were obtained by the author in [1]:

THEOREM 1. $(L^1, L \log^+ L)_{\theta,q;K} = A^{\theta q}$, $0 < \theta < 1$, $1 \leqq q \leqq \infty$.

COROLLARY 1.1. $(L^1, L \log^+ L)_{\theta,1;K} = L(\log^+ L)^\theta$, $0 < \theta < 1$.

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