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DIRICHLET FINITE BIHARMONIC FUNCTIONS ON THE POINCARÉ N-BALL

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On a Riemannian manifold R, let $\Delta = d\delta + \delta d$ be the Laplace-Beltrami operator. By definition, a sufficiently smooth function u on R is harmonic (biharmonic) if $\Delta u = 0$ ($\Delta^2 u = 0$). Denote by D the class of those functions f on R for which $D(f) = \int_R df \wedge * df$ is well defined and finite.

By the Poincaré N-ball we mean the ball

$$B_\alpha^N = \{x = (x^1, \ldots, x^N) \mid |x| < 1\},$$

α constant, endowed with the Poincaré-type metric

$$ds_\alpha = \lambda(x)\,|dx|, \qquad \lambda(x) = (1 - |x|^2)^\alpha.$$

The first purpose of this paper is to determine those values of the parameter α for which the class $H^2D(B_\alpha^N)$, $N \geqq 3$, of Dirichlet finite nonharmonic biharmonic functions on B_α^N is nonvoid. In Sario-Wang [3] it was proved that $H^2D(B_\alpha^N) \neq \emptyset$ for $N = 3$ if and only if $\alpha > -3/5$, and the question was raised whether the same is true for every N if and only if $\alpha > -3/(N + 2)$. We show that this is indeed so if $3 \leqq N \leqq 6$. However, quite unexpectedly, for $N > 6$ it turns out that $H^2D(B_\alpha^N) \neq \emptyset$ if and only if $\alpha \in (-3/(N + 2), 5/(N - 6))$.

The above result has interesting applications to the classification theory. Let Q be the class of quasiharmonic functions u, defined by $\Delta u = 1$, and denote by QD the subclass of Dirichlet finite functions in Q. The classes O_G, O_{QD}, and O_{H^2D} of Riemannian manifolds without Green's functions, QD-functions, and H^2D-functions, respectively, have the following properties:

(i) For every N, the classes O_{QD} and O_{H^2D} decompose the totality of Riemannian N-manifolds into three nonempty disjoint subclasses.

(ii) For every N, the class $O_G - O_{H^2D}$ is nonvoid.

(iii) For $N > 6$, the classes O_G and O_{H^2D} decompose the totality of Riemannian N-manifolds into four nonempty disjoint subclasses.

(iv) The unit N-ball with the natural metric $(1 - |x|^2)\,|dx|$ belongs to O_{H^2D} if and only if $N > 10$.

The proofs will appear in [1].

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