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THE GREEN FUNCTION OF A LINEAR DIFFERENTIAL EQUATION WITH A LATERAL CONDITION

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Let E be a Banach space. We consider systems of the form

$$\text{(L)} \quad L[y] \equiv y' + Ay = f, \qquad \text{(F)} \quad F[y] = c$$

where $y \in \mathscr{C}^{(1)}([a, b], E)$, $f \in \mathscr{C}([a, b], E)$, $A \in \mathscr{C}([a, b], L(E))$, $F \in L[\mathscr{C}([a, b], E), E]$ and $c \in E$. When the system has one and only one solution, for any $f \in \mathscr{C}([a, b], E)$ and $c \in E$, we show that it has a Green function, that is, a function $G:[a, b] \times [a, b] \to L(E, E'')$ such that $y \in \mathscr{C}^{(1)}([a, b], E)$ is the solution of $L[y] = f$ and $F[y] = 0$ if and only if $y(t) = \int_a^b G(t, s)f(s)\, ds$. We exhibit the relations between G, A and F. (F) is called a *lateral condition*; initial conditions and boundary conditions are particular instances of lateral conditions. The construction of G uses a Riemann-Stieltjes integral representation for F, given in §1.

1. **Analytic preliminaries.** We consider always vector spaces over the complex field $\boldsymbol{C}$, but all results are valid for real vector spaces.

1. Given an interval $[a, b]$ of the real line, a *division* of $[a, b]$ is a finite sequence $d: t_0 = a < t_1 < \cdots < t_n = b$. We write $|d| = n$ and $\Delta d = \sup\{|t_i - t_{i-1}| \mid i = 1, 2, \ldots, |d|\}$; D denotes the set of all divisions of $[a, b]$.

2. Let X, Y be Banach spaces; given $\alpha:[a, b] \to L(X, Y)$ and $d \in D$ we define

$$SV_d[\alpha] = \sup\left\{ \left\| \sum_{i=1}^{|d|} [\alpha(t_i) - \alpha(t_{i-1})]x_i \right\| \,\middle|\, x_i \in X, \|x_i\| \leqq 1 \right\}$$

and $SV[\alpha] = \sup\{SV_d[\alpha] \mid d \in D\}$.

We say that α is of *bounded semivariation*, and we write

$$\alpha \in SV([a,b], L(X, Y)),$$

if $SV[\alpha] < \infty$ (see for instance [**D**] and [**B-K**]).

PROPOSITION 1. *Given $\alpha \in SV([a, b], L(X, Y))$ and $f \in \mathscr{C}([a, b], X)$, there exists $F_\alpha[f] = \int_a^b d\alpha(t) \cdot f(t) = \lim_{\Delta d \to 0} \sum_{i=1}^{|d|} [\alpha(t_i) - \alpha(t_{i-1})] \cdot f(\xi_i) \in Y$, where $\xi_i \in [t_{i-1}, t_i]$, $i = 1, 2, \ldots, |d|$. We have $\|F_\alpha[f]\| \leqq SV[\alpha]\|f\|$ and hence*

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