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# LINK MANIFOLDS AND PERIODICITY

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1. **Introduction.** This research was motivated by a number of considerations not the least of which was an interesting phenomenon of periodicity observed by Alan Durfee [**1**]. Letting $M_k$ denote the Brieskorn variety $V(Z_0^k + Z_1^2 + Z_2^2 + \cdots + Z_n^2) \cap S^{2n+1}$ for $n \geqq 3$, odd and $k = 1, 2, 3, \ldots$, he finds that $M_k \simeq M_{k+8}$ where $\simeq$ denotes diffeomorphism. It was pleasant to speculate on possible geometric explanations for this regularity.

We examine this in the context of transformation groups. $M_k$ is an $O(n-1)$-manifold with three orbit types and orbit space $D^4$. The fixed point set corresponds to a torus link in $S^3 = \partial D^4$. Extending results of Hirzebruch and Erle [**2**] to the case of links we obtain a classification for these "link manifolds" in terms of invariants of links. These results are stated in §§2 and 3. §4 discusses periodicity. Proofs will appear elsewhere.

2. **Invariants of links.** Given a link $L \subset S^3$ together with assigned orientations for its components, one may form an oriented surface $F \subset S^3$ such that $\partial F = L$ and $F$ induces the given orientation on $L$. The Seifert pairing is a bilinear map $\theta: H_1(F) \times H_1(F) \to Z$ given by $\theta(x, y) = l(i_* x, y)$ where $i_*$ denotes "the push off of $F$ in the direction of the positive normal" and $l(\ ,\ )$ is the linking number in $S^3$. One then defines an even, symmetric bilinear form $f: H_1(F) \times H_1(F) \to Z$ by $f(x, y) = \theta(x, y) + \theta(y, x)$. It turns out that the signature of $f$ is an invariant of link type. In fact, it is identical with the Murasugi signature of the link [**3**]. We denote it by $\sigma(L)$. Let $Af: H_1(F) \to \mathrm{Hom}(H_1(F), Z)$ be the adjoint map $Af(x)(y) = f(x, y)$ and let $G(L) = \text{cokernel } (Af)$. Then $G(L)$ is also an invariant of $L$; it is the first homology group of the double branched cover of $S^3$ with branch set $L$.

Let $\tau G(L)$ be the torsion subgroup. Then there is a quadratic form $q(L): \tau G(L) \to Q/Z$. The equivalence class of this form is an invariant of $L$; it may be identified with the linking form of the double branched cover.

3. $O(n)$**-manifolds.**

DEFINITION. A *link-manifold* $M^{2n+1}$ is a smooth closed manifold admitting a smooth action of the orthogonal group $O(n)$ such that all isotropy subgroups are conjugate to $O(n)$, $O(n-1)$ or $O(n-2)$, the orbit space is diffeomorphic to $D^4 = \{x \in R^4 \mid \|x\| \leqq 1\}$ and the fixed point set in $M$ corresponds to a link $L \subset S^3 = \partial D^4$.

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