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ON THE DISCRETE REPRESENTATIONS OF THE GENERAL LINEAR GROUPS OVER A FINITE FIELD

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ABSTRACT. In this note we present a construction for a distinguished representation in the discrete series of $GL_n(F)$, F a finite field. This is used in describing explicitly Brauer's lifting of the identity representation of $GL_n(F)$.

Let V be a vector space of dimension $n \geqq 2$ over a finite field F with q elements. Let W_F be the ring of Witt vectors associated to F (see Serre [**4**,I, §6]) and K_F its field of fractions. Denote by $x \to \bar{x}$ the canonical ring homomorphism $W_F \to F$ and by $y \to \tilde{y}$ the canonical multiplicative homomorphism $F^* \to W_F$ such that $\tilde{y}^- = y$ for $y \in F^*$.

Our purpose is to construct explicitly a free W_F-module $D(V)$ associated canonically to V, which regarded as a representation of $GL(V)$ belongs to the discrete series, i.e. its character is a cusp form on $GL(V)$. We could call $D(V)$ the distinguished representation of the discrete series of $GL(V)$.

The construction is as follows (the details will appear elsewhere). Let X be the set of all sequences $(A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_{n-1})$ of affine subspaces of V ($\dim A_i = i$) which are away from the origin, i.e. $0 \notin A_{n-1}$. Let $\mathscr{F}$ be the set of all functions $f: X \to W_F$. Consider the subset $\mathscr{F}' \subset \mathscr{F}$ consisting of all f's satisfying

(1) Given any fixed sequence $(A_0 \subset A_1 \subset \cdots \subset A_{i-1} \subset A_{i+1} \subset \cdots \subset A_{n-1})$ of affine subspaces of V away from the origin and a variable A_i between A_{i-1} and A_{i+1} away from the origin (there are q choices for A_i if $i = 0, n-1$ and $q+1$ choices if $0 < i < n-1$), we have

$$\sum_{A_i} f(A_0 \subset A_1 \subset \cdots \subset A_{i-1} \subset A_i \subset A_{i+1} \subset \cdots \subset A_{n-1}) = 0.$$

Define $\mathscr{F}'_{-1}$ as the set of all $f \in \mathscr{F}'$ satisfying the homogeneity condition

(2) $f(\lambda A_0 \subset \lambda A_1 \subset \cdots \subset \lambda A_{n-1}) = \tilde{\lambda}^{-1} f(A_0 \subset A_1 \subset \cdots \subset A_{n-1})$, $\forall \lambda \in F^*$.

It is clear that $\mathscr{F}$, $\mathscr{F}'$, $\mathscr{F}'_1$ are finitely generated free W_F-modules. Define a W_F-linear map $t: \mathscr{F} \to \mathscr{F}$ by the formula

(3) $(tf)(A_0 \subset A_1 \subset \cdots \subset A_{n-1}) = (-1)^{n-1} \sum f(A'_0 \subset A'_1 \subset \cdots \subset A'_{n-1})$, where the sum is extended over all $(A'_0 \subset A'_1 \subset \cdots \subset A'_{n-1})$ in X such that $A'_0 \in A_{n-1} - A_{n-2}$, $A'_1 \| 0A_0$, $A'_2 \| 0A_1, \ldots, A'_{n-1} \| 0A_{n-2}$ (observe that once A'_0 is chosen, the A'_i's for $i > 0$ are automatically determined so that

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