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INVARIANT SUBSPACES OF HARDY CLASSES ON INFINITELY CONNECTED PLANE DOMAINS[1]

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Let C be the complex plane, Ce the extended plane, $\Delta(a, r)$ the open disk of radius r centered at a, R a Riemann surface and $H^p(R)$ Hardy class H^p of R (cf. [5, pp. 9–12]). A now classical theorem of Beurling states that the closed subspaces of $H^2(\Delta(0, 1))$ invariant under multiplication by z are exactly the subspaces V of the form $V = I \cdot H^2(\Delta(0, 1))$, where I is an inner function determined up to multiplication by a constant of modulus 1 by V[1]. Analogous theorems hold for $H^p(R)$, where R is the interior of a compact bordered Riemann surface and $1 \leqq p \leqq \infty$. (If $p = \infty$, the proper topology for V to be closed in is either the β or bounded weak-star topology of Buck [2], or else the weak-star topology.) (Cf. [3], [4], [10], [13].) We have generalized these theorems to $H^p(R)$, where R is a certain type of infinitely connected plane domain.

Before stating our generalization, we must make several definitions. A *locally analytic modulus*, or *l.a.m.*, is a real valued function g on R such that for each simply connected open subset U of R, there exists f analytic on U such that $g = |f|$. The l.a.m. g is *inner* if $\log g = G + S$, where G is a sum of Green's functions and S is a singular harmonic function in the sense of Parreau ([8], cf. also [5, p. 7]). If $R = \Delta(0, 1)$, an analytic function I is inner in the usual sense [6, pp. 61–68] if and only if the l.a.m. $|I|$ is inner.

R is a Blaschke region in case $R \subseteq C$ and R is of the form $Ce \sim \bigcup\{A(i): 0 \leqq i < \infty\}$ (or, $Ce \sim \bigcup\{A(i): 0 \leqq i \leqq n\}$) where the $A(i)$ are pairwise disjoint continua such that $Ce \sim A(i)$ is connected for each i. In addition, there must exist an integer n such that the $A(i)$ cluster only on $\bigcup\{A(i): 0 \leqq i \leqq n\}$, and a sequence $a(i) \in A(i)$, $i \geqq n + 1$, such that $\sum(G(a(i), z): n + 1 \leqq i < \infty) < \infty$. Here $G(a, z)$ is the Green's function for $Ce \sim \bigcup\{A(i): 0 \leqq i \leqq n\}$. Voichick first studied this class of plane regions [13]. We call them Blaschke regions because the prototype of

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