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ON THE SOLVABILITY OF SEMILINEAR OPERATOR EQUATIONS AND ELLIPTIC BOUNDARY VALUE PROBLEMS

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Let L be a bounded linear Fredholm mapping of index zero, mapping a Banach space X into a Banach space Y. Then necessary and sufficient conditions for the solvability of the operator equation $Lu = f$ for $f \in Y$ are well known. However the same satisfactory state of affairs does not hold for semilinear operator equations in which a compact nonlinear operator Nu is added to the right hand side of $Lu = f$. Indeed, most recent approaches to this problem attempt to place enough restrictions on the operator N so that $L - N$ is surjective.

A similar state of affairs holds in the case of elliptic boundary value problems defined on bounded domains $\Omega \subset R^N$. Recently, however, in the special case of the semilinear Dirichlet problem,

$$(1)\qquad \Delta u + \lambda_1 u + h(x, u) = f, \qquad u|_{\partial\Omega} = g(x), \qquad |h(x, u)| \leqq M,$$

where λ_1 denotes the smallest eigenvalue of the Laplacian Δ on Ω, Landesmann and Lazar [1] found interesting necessary and sufficient conditions for solvability that reduce to the usual orthogonality restrictions when $h(x, u) \equiv 0$. In [2], Nirenberg generalized the sufficiency conditions of [1] to semilinear operator equations of the form $Lu = Nu + f$, where L is a Fredholm map of index $p \geqq 0$ of a Banach space X into itself, and N is a compact map of X into itself. He also applied his result to the solvability of a large class of semilinear elliptic boundary value problems. However the conditions obtained by Nirenberg are not necessary in general, and in addition require the computation of topological invariants for their applicability. See also [3].

It is the purpose of this paper to provide computable *necessary and sufficient conditions* for the solvability of a class of semilinear problems intermediate between those of [1] and [2]. The conditions we obtain, on the one hand, can be considered as extensions of orthogonality to a nonlinear context; and on the other hand, are independent of topological considerations.

1. **The Hilbert space problem.** Let H be a Hilbert space over the reals,

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