BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 78, Number 4, July 1972

LOCALIZATION AND COMPLETION

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Communicated by Alex Rosenberg, December 27, 1971

Let I be an injective right R-module and E its ring of endomorphisms. The functor

$$\mathrm{Hom}_R(-, I): \mathrm{Mod}\, R \to (E\, \mathrm{Mod})^{\mathrm{op}}$$

has a right adjoint $\mathrm{Hom}_E(-, I)$, giving rise to a triple (standard construction) (S, η, μ), where

$$S = \mathrm{Hom}_E(\mathrm{Hom}_R(-, I), I).$$

Let $Q \to S$ be the equalizer of the pair of maps $\eta S, S\eta: S \to S^2$, then Q is the bext co-approximation of S by an idempotent triple [**2**].

$Q(M)$ is called the *localization* or *module of quotients* of M at I, and $Q(R)$ is a ring, the *ring of quotients*. $Q(M)$ may also be described as the divisible hull of M made torsionfree, in the *torsion theory* obtained from I (see e.g. [**4**]). In this torsion theory one calls M *torsion* if $\mathrm{Hom}_R(M, I) = 0$, *torsionfree* if M is isomorphic to a submodule of a power of I, *divisible* if $I(M)/M$ is torsionfree, where $I(M)$ is the injective hull of M.

The endofunctor Q of $\mathrm{Mod}\, R$ is always left exact. It is isomorphic to the identity functor if and only if it may be obtained from

$$I = \mathrm{Hom}_R(F, \boldsymbol{Q}/\boldsymbol{Z}),$$

where F is a free left R-module. Q is isomorphic to $(-) \otimes_R Q(R)$ if and only if it may be obtained from $I = \mathrm{Hom}_R(F, \boldsymbol{Q}/\boldsymbol{Z})$, where $R \to F$ is an epimorphism of rings and F is a flat left R-module [**4**], [**7**], [**10**]. Q is exact if and only if $I(M)/M$ is divisible for all torsionfree divisible modules M (compare with [**3**]), and then $Q(M) \cong M \otimes_R Q(R)$ for every finitely presented module M. We also note that every divisible module is injective if and only if I has zero singular submodule.

The I-adic topology on M is defined by taking as fundamental open neighborhoods of zero all kernels of homomorphisms $M \to I^n$, where n is finite. This topology and its relation to the usual P-adic topology has been discussed in [**5**].

The torsion submodule of M is open in the I-adic topology if and only if $\mathrm{Hom}_R(M, I)$ is a finitely generated E-module, or, equivalently, $\mathrm{Hom}_R(M, I^n)$ is a principal $\mathrm{End}_R(I^n)$-module for some finite n. These

AMS 1970 *subject classifications.* Primary 16A08; Secondary 18C15.