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ON THE EMBEDDING PROBLEM FOR NONSOLVABLE GALOIS GROUPS OF ALGEBRAIC NUMBER FIELDS: REDUCTION THEOREMS

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Let k be a field, K/k a finite Galois extension, G a finite group isomorphic to $\bar{G} = \mathrm{Gal}(K/k)$, $\gamma:\bar{G} \to G$ an isomorphism and $\Sigma: 1 \to N \to_{\iota} E \to_{\varepsilon} G \to 1$ an exact sequence of finite groups. The embedding problem

$$P = P(K/k, \Sigma, \gamma)$$

is to construct an extension L/K such that L/k is Galois, and such that there exists an isomorphism $\beta:\bar{E} \to E$, where $\bar{E} = \mathrm{Gal}(L/k)$, such that $\gamma \cdot \mathrm{Res}_{L/K} = \varepsilon\beta$. L is called a solution field, β a solution isomorphism, and the pair (L, β) a *solution*, to P. At times we only require β to be monomorphic; in such a context (L, β) is called an *improper* solution, and if β is epi, (L, β) is a *proper* solution.

1. **Reduction to solvable groups and split extensions.** Let $1 \to N \to_{\iota} E \to_{\varepsilon} G \to 1$ be an exact sequence of groups, and let U be a subgroup of E such that $U \cdot \iota(N) = E$. Let E^* be the semidirect product (U, N), where the action of U on N is given by $n^u = \iota^{-1}(u^{-1}\iota(n)u)$, for $n \in N, u \in U$. Let the mapping $\eta: E^* \to E$ be defined by $\eta((u, n)) = u\iota(n)$. One verifies easily that η is an epimorphism with kernel $U \cap \iota N$, and the diagram

$$\begin{array}{ccccccccc} 1 & \to & N & \xrightarrow{\iota^*} & E^* & \xrightarrow{\varepsilon^*} & U & \to & 1 \\ & & \| & & \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\varepsilon} & & \\ 1 & \to & N & \xrightarrow[\iota]{} & E & \xrightarrow[\varepsilon]{} & G & \to & 1 \end{array}$$

commutes and has exact rows, where $\varepsilon^*((u, n)) = u$ for $(u, n) \in E^*$, $\iota^*(n) = (1, n)$.

Let an embedding problem $P = P(K/k, \Sigma, \gamma)$ be given and let U be as above. We define the embedding problem $P_1 = P(K/k, \Sigma_1, \gamma)$ where Σ_1 is the sequence $1 \to \iota^{-1}(U \cap \iota N) \to_{\iota} U \to_{\varepsilon} G \to 1$. Suppose P_1 has a solution (L_1, β_1). We then define the embedding problem

$$P_2 = P(L_1/k, \Sigma_2, \beta_1)$$

where Σ_2 is $1 \to N \to_{\iota^*} E^* \to_{\varepsilon^*} U \to 1$. Suppose P_2 has a solution (L_2, β_2).

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