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A REMARK ON STRONG PSEUDOCONVEXITY FOR ELLIPTIC OPERATORS

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The purpose of this note is to give a kind of intrinsic characterization of second order elliptic operators.

Let $p(x, D)$ be an mth order elliptic operator defined on an open subset U of $\boldsymbol{R}^n$. Let $p_m(x, \xi)$ be its leading symbol. Let φ be a smooth function on U with the property that grad $\varphi \neq 0$ when $\varphi = 0$. The hypersurface, $\varphi = 0$, is said to be *strongly pseudoconvex* at a point $x \in \varphi^{-1}(0)$ if

$$(1) \qquad \sum \frac{\partial^2 \varphi}{\partial x_j \partial x_k} p_m^{(j)}(x, \xi)\overline{p_m^{(k)}(x, \xi)} + \tau^{-1} \operatorname{Im} \sum p_{m,k}(x, \xi)\overline{p^{(k)}(x, \xi)} > 0,$$

for all $\xi = \eta + i\tau$ grad φ, where $\eta \in \boldsymbol{R}^n$ and $0 \neq \tau \in \boldsymbol{R}$, satisfying the equations:

$$(2) \qquad p_m(x, \xi) = 0 = \sum p_m^{(j)}(x, \xi)\, \partial\psi/\partial x_j.$$

(See Hörmander [**2**, Chapter 8].)

If p is a second order operator and its leading symbol is real, then, for $\eta, N \in \boldsymbol{R}^n$ not multiples of each other, the equations

$$(3) \qquad p_2(x, \eta + \tau N) = 0 = \sum p_2^{(j)}(x, \eta + \tau N)N_j$$

have no solutions, so condition (1) is satisfied trivially. This proves

PROPOSITION 1. *If $p(x, D)$ is second order and its leading symbol is real, then every hypersurface is strongly pseudoconvex.*

In this note we will prove a result in the other direction, namely,

PROPOSITION 2. *If $n \geqq 3$ and every hypersurface is strongly pseudoconvex then $p(x, D)$ is second order.*

REMARK 1. If there exist vectors η, N satisfying (3) it is easy to construct a φ, with grad $\varphi(x) = N$, violating (1). Therefore for every surface with normal, N, at x to be strongly pseudoconvex at x it is necessary and sufficient that there be no η, N satisfying (3). Hence Proposition 2 can be reformulated as a simple algebraic assertion, namely,

PROPOSITION 3. *Let $p(\zeta)$, $\zeta \in \boldsymbol{C}^n$, be a homogeneous polynomial of degree m. Assume $n \geqq 3$, and assume p satisfies the following conditions:*

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