BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 78, Number 3, May 1972

VALUES IN DIFFERENTIAL GAMES

BY ROBERT J. ELLIOTT AND NIGEL J. KALTON[1]

Communicated by James Serrin, October 21, 1971

1. **Introduction.** Two person zero-sum differential games can be considered as control problems with two opposing controllers or players. One player seeks to maximize and one to minimize the pay-off function. The greatest pay-off that the maximizing player can force is termed the lower value of the game and similarly the least value which the minimizing player can force is called the upper value. Our objective is to determine conditions under which these values coincide.

In the case of two person zero-sum matrix games, von Neumann showed that if the players are allowed "mixed strategies," i.e., probability measures over the pure strategies, then the values of the game will coincide. By analogy, the authors in collaboration with L. Markus [**1**] introduced relaxed controls into differential game theory; for a full discussion of relaxed controls the reader is referred to [**1**].

In this announcement, we define notions of strategy and values and relate these to the approaches adopted by Fleming [**3**], [**4**] and Friedman [**5**]. Using relaxed controls we are able to show that if the "Isaacs condition" (3) is satisfied then the upper and lower values are equal, so that the game has value. In particular, if the players are allowed relaxed controls then the game always has value. Detailed proofs of these results will appear in a later publication [**2**].

2. **Notation.** A differential game G played by two players J_1 and J_2 for the fixed time interval $I = [0, 1]$ is considered. At each time $t \in I$, J_1 picks an element $y(t)$ from a compact metric space Y and J_2 picks $z(t)$ from a similar space Z in such a way that the functions $t \to y(t)$ and $t \to z(t)$ are measurable. The dynamics are given by the differential equation

$$\dot{x} = dx/dt = f(t, x, y(t), z(t)). \tag{1}$$

Here $x \in \boldsymbol{R}^m$ and $f: I \times \boldsymbol{R}^m \times Y \times Z \to \boldsymbol{R}^m$ is a continuous function. For simplicity of exposition we assume f satisfies constant Lipschitz conditions in t and x. Assuming $x(0) = 0$, the above conditions ensure that for any pair of functions $(y(t), z(t))$ there is a trajectory $x(t)$. At the end of the game a pay-off to J_1,

$$P(y, z) = g(x(1)) + \int_0^1 h(t, x(t), y(t), z(t))\, dt, \tag{2}$$

AMS 1969 *subject classifications*. Primary 9025.

[1] Both authors are indebted to the S.R.C. for Research Fellowships.