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LÖWENHEIM-SKOLEM AND INTERPOLATION THEOREMS IN INFINITARY LANGUAGES[1]

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Let L be a first-order finitary predicate language with equality. For each pair of infinite cardinals κ and λ with $\kappa \geqq \lambda$ we let $L_{\kappa\lambda}$ be the logic extending L which allows the conjunction ($\wedge$) and disjunction ($\vee$) of fewer than κ formulas and the simultaneous universal or existential quantification of fewer than λ variables. We set $L_{\infty\lambda} = \bigcup_\kappa L_{\kappa\lambda}$. The standard syntactical and semantical concepts are defined as usual (see [**1**], [**2**]). If θ is a sentence we write $\mathfrak{A} \models \theta$ to mean that θ is true on the model $\mathfrak{A}$. $\mathfrak{A} \equiv_{\kappa\lambda} \mathfrak{B}$ means that $\mathfrak{A}$ and $\mathfrak{B}$ have the same true sentences of $L_{\kappa\lambda}$. $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{A}_i$ are always used for models for L, and we follow the convention that their universes are A, B, A_i respectively. The cardinality of a set X is denoted by $|X|$. If L' is some other language, then $L'_{\kappa\lambda}$ is the corresponding infinitary logic built on L'. For ease in stating many of our results we assume, except in the last section, that L has only countably many nonlogical symbols. A detailed presentation of these and related results is in preparation for publication elsewhere.

1. $L_{\infty\omega}$ **and the Löwenheim-Skolem theorem.** One form of the downward Löwenheim-Skolem theorem for sentences of $L_{\omega_1\omega}$ can be stated as follows:

(A) If $\mathfrak{A} \models \theta$, then $\mathfrak{A}_0 \models \theta$ for some countable $\mathfrak{A}_0 \subseteq \mathfrak{A}$. The conclusion of (A) is quite weak; certainly the converse does not generally hold. One of our first goals is to define a notion of "almost all" such that the following biconditional holds for sentences of $L_{\omega_1\omega}$:

(B) $\mathfrak{A} \models \theta$ iff $\mathfrak{A}_0 \models \theta$ for almost all countable $\mathfrak{A}_0 \subseteq \mathfrak{A}$. More importantly, we also generalize (B) to apply to sentences of $L_{\infty\omega}$ (for which (A) usually fails). To do this we must first index the countable submodels of a model and define countable approximations to any sentence of $L_{\infty\omega}$.

Let κ be an uncountable cardinal. We define a filter D over $\mathscr{P}_{\omega_1}(\kappa)$, the countable subsets of κ, as follows:

DEFINITION. $X \subseteq \mathscr{P}_{\omega_1}(\kappa)$ belongs to D iff X contains some X' such that (i) for every $s \in \mathscr{P}_{\omega_1}(\kappa)$ there is some $s' \in X'$ such that $s \subseteq s'$ and (ii) X' is closed under unions of countable chains.

LEMMA. *D is a countably complete filter, and if $X_\xi \in D$ for all $\xi < \kappa$ then $\{s : s \in X_\xi$ for all $\xi \in s\} \in D$.*

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