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THE CONSTRUCTION OF AN ASYMPTOTIC CENTER WITH A FIXED-POINT PROPERTY[1]

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ABSTRACT. Given a bounded sequence $\{u_n: n = 1, 2, \ldots\}$ of points in a closed convex subset C of a uniformly convex Banach space, c_m denotes the point in C with the property that among all closed balls centered at points of C and containing $\{u_m, u_{m+1}, \ldots\}$ the one centered at c_m is of smallest radius. It is shown that the sequence $\{c_m: m = 1, 2, \ldots\}$ converges (strongly) to a point $c \in C$ called the asymptotic center of $\{u_n\}$ with respect to C. Further, for a class of mappings f of C into itself, which contains all nonexpansive mappings, $f(c) = c$ whenever an $x \in C$ exists such that $f^n(x) = u_n$, $n = 1, 2, \ldots$.

1. **Introduction.** Let C be a closed convex set in a uniformly convex Banach space X. (Recall that X is called uniformly convex if the modulus of convexity

$$\delta(\varepsilon) = \inf\{1 - \tfrac{1}{2}\|x + y\| : \|x\|, \|y\| \leqq 1, \|x - y\| \geqq \varepsilon\}$$

is positive in its domain of definition $\{\varepsilon : 0 < \varepsilon \leqq 2\}$.) Given a bounded sequence $\{u_n : n = 1, 2, \ldots\}$ in the set C, define

$$r_m(y) = \sup\{\|u_k - y\| : k \geqq m\} \quad (y \in X). \tag{1}$$

It is well known, and easily proved, that a unique point $c_m \in C$ exists such that

$$r_m(c_m) = \inf\{r_m(y) : y \in C\} = r_m. \tag{2}$$

Clearly $r_m \geqq r_{m+1}$ and $r_m \geqq 0$ for all $m = 1, 2, \ldots$ so that $\{r_m : m = 1, 2, \ldots\}$ converges to $r = \inf\{r_m : m = 1, 2, \ldots\}$. We note that if $r = 0$ then, as can be readily verified, the sequence $\{u_n\}$ converges.

2. **The asymptotic center.**

DEFINITION. If $\{c_m\}$ converges then $c = \lim c_n$ is called the asymptotic center of $\{u_n\}$ (with respect to C).

THEOREM 1. *With X, C and $\{u_n\}$ as above, the sequence $\{c_m\}$ converges. (Thus the asymptotic center c exists.)*

PROOF. If $r = 0$ then, as can be readily seen, $\{u_n\}$ is a Cauchy sequence and $\lim_{n\to\infty} u_n = \lim_{m\to\infty} c_m\,(=c)$. We may then assume that $r > 0$. Suppose now, for a contradiction, that $\{c_m\}$ fails to converge. Then an $\varepsilon > 0$ exists such that for any natural number N there are integers

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