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# $BV$-FUNCTIONS ON COMMUTATIVE SEMIGROUPS

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The concept of functions of bounded variation on a linearly ordered set has been generalized to a distributive lattice [**1**] and more recently to a semilattice (cf. [**3**], [**4**] and [**5**]). Here, using different techniques, we further extend this notion to commutative semigroups with identity and show that the $BV$-functions characterize the "abstract moment sequences" or what we call moment functions.

Let $S$ be a commutative semigroup with identity 1. A nontrivial homomorphism, which maps $S$ into the multiplicative semigroup of nonnegative real numbers not greater than 1, will be called an *exponential*. We will denote the set of all exponentials on $S$ by $\exp(S)$. Equipped with the topology of simple convergence, $\exp(S)$ is a compact Hausdorff space. We now formulate the *abstract moment problem*. Given a real-valued function $f$ on $S$, when does there exist a regular Borel measure $\mu_f$ on $\exp(S)$ such that $f(x) = \int_{e \in \exp(S)} e(x) d\mu_f(e)$ for all $x \in S$? The Stone-Weierstrass theorem implies the uniqueness of the representing measure (cf. [**2**]), when it exists. Thus using the terminology of [**6**], the abstract moment problem is completely determined. Those functions on $S$ which admit representing measures will be called *moment functions*.

The exponentials of the semigroup $N$ of nonnegative integers under addition can be identified with the closed unit interval $[0, 1]$ in a natural way. Hence, if $S = N$, the abstract moment problem reduces to the already solved little moment problem of Hausdorff and a real-valued function on $N$ is a moment function if and only if it is a moment sequence in the classical sense (cf. [**7**, p. 100]).

Our main result is then that the moment functions and $BV$-functions agree. The methods used provide a new proof of the classical characterization of moment sequences. Principal results on $BV$-functions contained in [**1**], [**3**] and [**4**] also follow in a new way.

Let $f$ be a real-valued function on $S$ and $x \in S$. The translate function $f_x$ of $f$ by $x$ is defined in the usual way by $f_x(y) = f(xy)$ for $y \in S$. Successive differences of $f$ can be defined inductively by

$$\Delta_0 f(\circ) \equiv f(\circ) \quad \text{and} \quad \Delta_n f(\circ; h_1, \ldots, h_n) = \Delta_{n-1}(f - f_{h_n})(\cdot; h_1, \cdot + \cdot, h_{n-1})$$

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