BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 77, Number 6, November 1971

ON EQUIDISTANT CUBIC SPLINE INTERPOLATION

BY I. J. SCHOENBERG

Communicated by Fred Brauer, May 14, 1971

Let n be a natural number and let $S[0, n]$ denote the class of cubic spline functions $S(x)$ defined in the interval $[0, n]$ and having the points $1, \cdots, n-1$ as knots. This means that the restriction of $S(x)$ to the inverval $(\nu, \nu+1)$ $(\nu=0, \cdots, n-1)$ is a cubic polynomial, and that $S(x) \in C^2[0, n]$. The most general element of $S[0, n]$ is evidently of the form

$$S(x) = \sum_0^3 a_\nu x^\nu + \sum_1^{n-1} c_\nu (x-\nu)_+^3, \tag{1}$$

and is seen to depend linearly on $n+3$ parameters. Here we have used the function $x_+ = \max(0, x)$.

The interpolatory properties of the elements of the class $S[0, n]$ have recently attracted considerable attention. The two main kinds of this so-called *cubic spline interpolation* are as follows.

1. *Natural cubic spline interpolation.* We are required to find $S(x) \in S[0, n]$ such as to satisfy the conditions

$$S(\nu) = f(\nu) \qquad (\nu = 0, \cdots, n), \tag{2}$$

$$S''(0) = S''(n) = 0. \tag{3}$$

2. *Complete cubic spline interpolation.* Here we are asked to find $\tilde{S}(x) \in S[0, n]$ so as to satisfy the conditions

$$\tilde{S}(\nu) = f(\nu) \qquad (\nu = 0, \cdots, n), \tag{4}$$

$$\tilde{S}'(0) = f'(0), \qquad \tilde{S}'(n) = f'(n). \tag{5}$$

The existence and unicity of the solutions of these problems is widely known. See [**1**], [**2**], or [**5**, Chapter 4] for illuminating discussions of these problems.

We may describe the solution of the first problem (2), (3) by the interpolation formula

$$f(x) = \sum_0^n f(\nu) L_\nu(x) + R, \tag{6}$$

where the fundamental functions $L_\nu(x)$ are elements of $S[0, n]$ and

AMS 1970 *subject classifications.* Primary 41A15.