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# ON THE EXISTENCE OF A CONTROL MEASURE FOR STRONGLY BOUNDED VECTOR MEASURES

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In this note we extend a theorem of Bartle, Dunford, and Schwartz [1] which states that for every countably additive measure defined on a $\sigma$-algebra there exists a positive "control measure" $\nu$ such that $\nu(E)\to 0$ if and only if $\|\mu\|(E)\to 0$, where $\|\mu\|$ is the semivariation of $\mu$. In this paper, $\mu$, which is defined on a ring $\Sigma$, is assumed to be finitely additive and strongly bounded ($s$-bounded) [8] (that is $\mu(E_i)\to 0$ whenever $\{E_i\}$ is a disjoint sequence of sets). The existing decomposition and extension theorems for vector measures can now be easily deduced by using the control measure. These applications will be presented in [2].

$\mathfrak{X}$ is a Banach space over the reals (the complex case is treated in a similar fashion); $S^*$ is the unit sphere in the conjugate space of $\mathfrak{X}$. $\sigma(\mathcal{E})$ denotes the $\sigma$-algebra generated by the class of sets $\mathcal{E}$. A $\delta$-ring is a ring of sets closed under countable intersections.

THEOREM 1. *Let $\Sigma$ be a ring of subsets of a set $S$. $\mu:\Sigma\to\mathfrak{X}$ is finitely additive and s-bounded if and only if there exists a positive finitely additive bounded set function $\nu$ defined on $\Sigma$ such that*

$$\lim_{\nu(E)\to 0} \mu(E) = 0$$

*and*

$$\nu(E) \leqq \sup\{\|\mu(F)\| : F \subseteq E,\ F \in \Sigma\}, \qquad E \in \Sigma.$$

SKETCH OF THE PROOF. First assume $\Sigma$ is an algebra. Let $T$ be the isometric isomorphism of $ba(S, \Sigma)$ onto $ba(S_1, \Sigma_1)$, where $\Sigma_1$ is the Stone algebra of all open-closed subsets of the compact totally disconnected Hausdorff space $S_1$ [4, IV.9]. $U$ is the isometric isomorphism between $ba(S_1, \Sigma_1)$ and $ca(S_1, \Sigma_2)$, where $\Sigma_2=\sigma(\Sigma_1)$.

We prove that $\{(UT)(x^*\mu):x^*\in S^*\}$ is uniformly countably additive on $\Sigma_2$. It suffices to show that $\{(UT)[(x^*\mu)^+]:x^*\in S^*\}$ is uniformly countably additive, where $x^*\mu=(x^*\mu)^+-(x^*\mu)^-$ is the Jordan

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