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INTEGRABILITY CONDITIONS FOR $\Delta u = k - Ke^{\alpha u}$ WITH APPLICATIONS TO RIEMANNIAN GEOMETRY

BY JERRY L. KAZDAN[1] AND F. W. WARNER[1]

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1. In this note we announce some integrability conditions for the equation $\Delta u = k - Ke^{\alpha u}$ on compact orientable Riemannian 2-manifolds (where Δ is the Laplacian), and we give some applications to problems in Riemannian geometry. Further results and details will appear elsewhere. We begin with a description of the geometry problem which led us to a study of the above equation. M, throughout, will denote a compact, connected, oriented, 2-dimensional manifold.

Problem. What are necessary and sufficient conditions on a sufficiently smooth (we shall restrict ourselves to C^∞ data here) function K on M for K to be the Gaussian curvature of some Riemannian metric on M?

If K is the Gaussian curvature of a Riemannian metric g with volume form ω on M, the only known global condition which K must satisfy is the Gauss-Bonnet formula

$$\int_M K\omega = 2\pi\chi(M), \tag{1}$$

where $\chi(M)$ is the Euler characteristic of M. Here $K\omega$ is called the curvature form of the metric g. One can rephrase the above question in terms of curvature forms, and in this case it turns out [9] that the condition $\int_M \Omega = 2\pi\chi(M)$ is not only necessary but also is sufficient for a two form Ω to be the curvature form of a Riemannian metric on M. As for the Gaussian curvature functions themselves, (1) seems only to impose certain sign requirements depending on the genus of M. Specifically, it seems natural to expect that a necessary and sufficient condition for a smooth function K to be the Gaussian curvature of a Riemannian metric on M is

(i) that K be positive somewhere if genus$(M) = 0$,

(ii) that K change sign, if not identically 0, if genus$(M) = 1$,

(iii) that K be negative somewhere if genus$(M) > 1$.

As a special case of (i), H. Gluck has recently shown [3] that K is a Gaussian curvature if K is strictly positive. His approach is to

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