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C^1 PARTITIONS OF UNITY ON NONSEPARABLE HILBERT SPACE

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The main result is that every Hilbert space admits continuously differentiable partitions of unity. We sketch a proof of the key proposition. Details will appear in [4].

Much more is known for separable Banach spaces. R. Bonic and J. Frampton [1] showed that if there are any nontrivial C^k (i.e., k continuous Fréchet derivatives) on E, a separable Banach space, then E admits C^k partitions of unity. Thus separable Hilbert space, l^n, for n an even integer, and c_0 admit C^∞ partitions of unity. C^∞ partitions of unity on separable l^2 were first constructed by James Eells; a proof appears in [2].

Let R^n be n-dimensional Euclidean space. Let

$$C^k_{1,M} = \left\{ f \,\middle|\, f \in C^k(R^n, R), \sup_{x \neq y} (\|D^k f(x) - D^k f(y)\| / \|x - y\|) \leqq M \right\}.$$

If A is a closed subset of R^n, call f a $C^k_{1,M}$ A-selecting function if $f \in C^k_{1,M}$, $0 \leqq f(x) \leqq 1$, $C^k_{1,M}(x) = 1$ if $x \in A$ and $f(x) = 0$ if $d(x, A) \geqq 1$. By smoothing out $\sup(0, 1 - d(x, A))$ we can always find a $C^k_{1,M}$ A-selecting function provided M is large enough. For $k = 0$, $f(x) = \sup(0, 1 - d(x, A))$ has smallest M namely 1. For $k = 1$ and 2, we have the following:

THEOREM 1. *Let* $A = \{x \mid x_i \leqq 0, \|x\| \leqq 1, i = 1, \cdots, n\}$. *Then if f is a* $C^2_{1,M}$ *A-selecting function,* $n > M^2 + 36M^4$.

COROLLARY 1. *The Whitney Extension Theorem fails for separable Hilbert space.*

THEOREM 2. *If A is a closed subset of H, any Hilbert space, then there exists a* $C^1_{1,4}$ *A-selecting function, f, and if* $g \in C^1_{1,4}(H, R)$, $g(x) = 1$ *for x in A and* $0 \leqq g(x) \leqq 1$, *then* $f(x) \leqq g(x)$.

The key to the proof of Theorem 2 is

PROPOSITION 1. *Theorem 2 is true if H is finite dimensional and $A = F$ a finite subset.*

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