BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 77, Number 5, September 1971

INVARIANT POLYNOMIALS AND CONJUGACY CLASSES OF REAL CARTAN SUBALGEBRAS[1]

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Communicated by M. Gerstenhaber, March 29, 1971

1. Introduction and statement of main result. For complex reductive Lie algebras the conjugacy of all the Cartan subalgebras by inner automorphisms plays a fundamental role in the structure and classification theory. The conjugacy theorem does not hold for real reductive Lie algebras, but each such algebra has only finitely many conjugacy classes of Cartan subalgebras. A complete classification of the conjugacy classes for all real simple Lie algebras was done by Kostant in 1955. (See Kostant [4], and Sugiura [7].) The proof of conjugacy of Cartan subalgebras in the complex case, requiring only the connectedness of the subset of regular semisimple elements [6, p. III-6, Theorem 2], suggests a relationship, in the real case, between the connected components of this subset and the conjugacy classes of Cartan subalgebras. Our purpose here is to establish such a relationship, using the invariant polynomials on the Lie algebra. We also show how this relates to the number of real zeros of polynomials.

Let $\mathfrak{g}_c$ be a complex reductive Lie algebra, and G_c any corresponding connected subgroup. We shall write $a \cdot x$ for the adjoint action of $a \in G_c$ on $x \in \mathfrak{g}_c$. Let J be the ring of all polynomials on $\mathfrak{g}_c$ invariant under the adjoint action of G_c. By a famous result of Chevalley, there exist homogeneous polynomials $u_1, u_2, \cdots, u_l$, where l is the rank of $\mathfrak{g}_c$, which are algebraically independent and generate J. (See Chevalley [1] and Helgason [2, pp. 429–434].)

$x \in \mathfrak{g}_c$ is called *semisimple* if ad x is a diagonalizable endomorphism, where ad x is the endomorphism of $\mathfrak{g}_c$ defined by ad $x \cdot y = [x, y]$ for any $y \in \mathfrak{g}_c$. x is called *regular* if its orbit under the action of G has maximal possible dimension. For any real form $\mathfrak{g}$ of $\mathfrak{g}_c$ let $G \subset G_c$ be the connected subgroup corresponding to $\mathfrak{g}$. G acts as a group of automorphisms on $\mathfrak{g}$, and two Cartan subalgebras of $\mathfrak{g}$ are said to be *conjugate* if one is transformed into the other by some automorphism in G.

AMS 1970 *subject classifications*. Primary 17B20, 17B40, 22E15; Secondary 12D16, 53A55.

Key words and phrases. Lie algebras, semisimple Lie groups, Cartan subalgebras, invariant polynomials.

[1] This work was supported by an N.S.F. Graduate Fellowship.