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SOME RELATIONS BETWEEN THE METRIC STRUCTURE AND THE ALGEBRAIC STRUCTURE OF THE FUNDAMENTAL GROUP IN MANIFOLDS OF NONPOSITIVE CURVATURE

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1. **Introduction.** Let $\hat{M}$ be a complete simply connected riemannian manifold of dimension n and sectional curvature $K \leqq 0$. Working with closed geodesically convex subsets $\emptyset \neq M \subset \hat{M}$, we use the fact (see [4] or [6]) that M is a topological submanifold of $\hat{M}$ of some dimension k, $0 \leqq k \leqq n$, with totally geodesic interior ${}^0M$ and possibly empty boundary ∂M. Note that M is star-shaped from every point, thus contractible, and in particular simply connected.

Consider a properly discontinuous group Γ of homeomorphisms of M that acts by isometries on ${}^0M$. If the elements of Γ satisfy the semisimplicity condition described below (automatic if $\Gamma\backslash M$ is compact), and if Σ is a solvable subgroup of Γ, then Theorem 1 exhibits a flat totally geodesic Σ-stable subspace $E \subset M$, complete in $\hat{M}$, such that Σ has finite kernel on E and $\Sigma\backslash E$ is compact. Thus Σ is an extension of a finite group by a crystallographic group of rank dim E. In particular,

(i) Σ is finitely generated,

(ii) if $\Sigma\backslash M$ is compact, then M is a complete flat totally geodesic subspace of $\hat{M}$, and

(iii) if $\Gamma\backslash M$ is a manifold, then the image of E in $\Gamma\backslash M$ is a compact totally geodesic euclidean space form.

Theorem 1 extends and unifies several results concerning the case where $M = \hat{M}$ and $\Gamma\backslash M$ is a compact manifold. Those results are the classical theorem of Preissmann [7] which says that if $K < 0$ then every nontrivial abelian subgroup of Γ is infinite cyclic, Byers' extension [2] of Preissmann's theorem to solvable subgroups of Γ, the case [10] where the elements of Σ are bounded isometries of M, the case [11] where Σ is central in Γ, and the case [11] where Γ is nilpotent. Theorem 1 was known [9] in the case where M is riemannian symmetric and $\Gamma\backslash M$ is compact. The case where $M = \hat{M}$ and $\Gamma\backslash M$ is

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