A CHARACTERIZATION THEOREM FOR CELLULAR MAPS

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Communicated by Steve Armentrout, March 26, 1970

Introduction. The main result of this paper is that a mapping f of the n-sphere ∂B^{n+1}, $n \neq 4$, onto itself is cellular if and only if f has a continuous extension which maps the interior of the $n+1$ ball B^{n+1} homeomorphically onto itself. Since a map of a 2-sphere onto itself is cellular if and only if it is monotone, this theorem extends a result of Floyd and Fort [6], who prove the corresponding theorem for monotone maps on a 2-sphere.

Preliminaries. A compact mapping $f: M^n \to X$ is cellular if for each $x \in X$, there is a sequence $C_1, C_2, \cdots$ of topological n-cells such that $f^{-1}(x) = \bigcap_{i=1}^{\infty} C_i$ and $C_{i+1} \subset \operatorname{Int} C_i$. If X is a topological space, $H(X)$ is the group of all homeomorphisms of X onto itself. Edwards and Kirby showed that for any compact manifold M, $H(M)$ is locally contractible and therefore uniformly locally arcwise connected. It was shown [7] that any mapping of a manifold onto itself which can be uniformly approximated by homeomorphisms is cellular. (See also [4].) Armentrout ($n=3$) [1] and Siebenmann ($n \geqq 5$) [10] have proven that any cellular mapping of a manifold onto itself can be uniformly approximated by homeomorphisms.

LEMMA 1. *Suppose $f: \partial B^n \to \partial B^n$ can be approximated by homeomorphisms. Then f can be extended to a map which is a homeomorphism on the interior of B^n.*

PROOF. Since f can be uniformly approximated by homeomorphisms and $H(\partial B^n)$ is uniformly arcwise connected, there is an arc Φ such that $\Phi_1 = f$ and $\Phi_t \in H(\partial B^n)$, for $0 \leqq t < 1$. Each point of B^n can be represented in the form tx, where $x \in \partial B^n$ and $0 = t = 1$. We define $F: B^n \to B^n$ by $F(tx) = t\Phi_t(x)$, for all $x \in \partial B^n$. We note that F is continuous, extends f and is a homeomorphism when restricted to the interior of B^n.

Therefore, if $n \neq 4$ and $f: \partial B^{n+1} \to \partial B^{n+1}$ is cellular f can be extended to a map which is a homeomorphism on the interior of B^{n+1}.

AMS 1970 *subject classifications.* Primary 5460; Secondary 5701.

Key words and phrases. Cellular maps, monotone maps, extending mappings, UV^{∞} maps.

[1] This paper represents a portion of the author's Ph.D. thesis, written under the direction of Louis F. McAuley to be presented to the faculty of the State University of New York at Binghamton.