EXTREME POINTS OF THE SET OF UNIVALENT FUNCTIONS

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Let S be the usual set of analytic, normalized, univalent functions on the unit disk. A function belonging to S is called an *extreme point of* S if it cannot be written as a proper convex combination of two other members of S. (See [1, p. 439].) For example the Koebe function is an extreme point of S, a fact that follows, for instance, from the unique maximal property of the second coefficient. We shall let E denote the set of extreme points of S.

The set of all analytic functions on the unit disk is a locally convex linear topological space, and S is a compact subset [2, p. 217]. Therefore the conclusion of the Krein-Milman theorem [1, p. 440] applies. Namely, $S \subset \mathrm{Cl}(\mathrm{co}\, E)$. In other words every function in S is the limit, uniform on each compact subset of the disk, of a suitable sequence of convex combinations of extreme points. Thus, the determination of of E should provide a tremendous amount of information about S. For example, the Bieberbach conjecture for the functions of E implies the full conjecture. In fact any continuous linear functional achieves its maximum real part, maximum modulus, etc. on S at a point of E. These statements follow from Lemma 2 [1, p. 439] and Lemma 3 [1, p. 440]. Although we have been unable to characterize the functions belonging to E, the theorem below provides a very simple necessary condition on the range of such a function. It is believed that the techniques used here can be applied further to refine this result.

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LEMMA 1. *Let $f \in S$. Suppose there is a function ϕ, analytic on range f but not of the form $\phi(w) = aw + b$, and two complex numbers, α and β, such that*

$$(\phi(w_1) - \phi(w_2)/(w_1 - w_2) \neq \alpha, \quad (\phi(w_1) - \phi(w_2))/(w_1 - w_2) \neq \beta$$

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