BAER SUBPLANES AND BLOCKING SETS

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A *blocking set* S in a projective plane π is a subset of the points of π such that every line of π contains at least one point of S and at least one point which is not in S. Denoting the number of points in S by $|S|$ our main result, obtained by purely combinatorial means, is the following: If π is finite of square order, say m^2 then $|S| \geqq m^2+m+1$ and if $|S| = m^2+m+1$ then the points of S are the points of a subplane of π of order m (a Baer subplane). In this connection we first of all prove the following

THEOREM. *Baer subplanes form blocking sets.*

PROOF. Suppose π is a plane of order m^2 which contains a subplane S of order m. Since any line of π contains at most $m+1$ points of S we have that every line of π contains at least one point which is not in S. Let l be any line of π and P be any point of l which is not in S. Then there is at most one line of S through P, S being a subplane. Also since any two points of π are connected by a unique line, the m^2+m+1 points of S are contained in the m^2+1 lines of π through P. If l contained no point of S, the lines of π through P would account for at most $(m+1)+(m^2-1)\cdot 1 = m^2+m$ points of S. Thus l must contain at least one point of S establishing our theorem.

We now proceed to the main result. π denotes a plane of order n and S is a blocking set in π. $S-l$ denotes all those points P such that P is contained in S but not in l, and $|S-l|$ means the number of such points P; similarly for $l-S$, $|l-S|$.

LEMMA 1. *No line of π contains more than $|S|-n$ points of S.*

PROOF. Let l be any line of π and suppose l contains exactly t points of S. Since S is a blocking set there is at least one point R in $l-S$. There are n lines of π through R besides l, each containing at least one point of S. Thus always $|S| \geqq t+n$.

LEMMA 2. *Let a objects be packed into b boxes such that each box contains at least one object, with $b \leqq a < 2b$. Define a function f on the objects X as follows:*

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