APPLICATIONS OF RADON-NIKODÝM THEOREMS TO MARTINGALES OF VECTOR VALUED FUNCTIONS

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The purpose of this note is to announce some new martingale convergence theorems derived as consequences of the Radon-Nikodým theorems, theorems for vector measures of Métivier [7] and Rieffel [8]. The results announced here contain theorems of Scalora [9], Chatterji [1] and [2], A. and C. Ionescu-Tulcea [5] and Métivier [7]. Throughout this note, unless explicitly mentioned otherwise, (Ω, Σ, μ) is a fixed finite measure space, $\mathfrak{X}$ is a Banach space, $L^p(\mu, \mathfrak{X})$, $(1 \leqq p < \infty)$ is the space of all strongly measurable $\mathfrak{X}$-valued (equivalence classes of) functions f on Ω satisfying $\|f\|_p = (\int_\Omega \|f\|^p d\mu)^{1/p} < \infty$. If T is a directed set, $\{B_\tau, \tau \in T\}$ is an increasing net of sub-σ-fields (i.e. $\tau_1 \leqq \tau_2$ implies $B_{\tau_1} \subset B_{\tau_2}$), then $\{f_\tau, B_\tau, \tau \in T\}$ is a martingale in $L^p(\mu, \mathfrak{X})$ $(1 \leqq p < \infty)$ if $f_\tau \in L^p(\mu, \mathfrak{X})$, f_τ is measurable relative to B_τ, and $\int_E f_{\tau_2} d\mu = \int_E f_{\tau_1} d\mu$ if $\tau_2 \geqq \tau_1$ and $E \in B_{\tau_1}$.

1. A characterization of mean convergent martingales in $L^p(\mu, \mathfrak{X})$. This section is devoted to characterizing mean convergent martingales in $L^p(\mu, \mathfrak{X})$. Basic to this section is the following

DEFINITION. A martingale $\{f_\tau, B_\tau, \tau \in T\}$ in $L^p(\mu, \mathfrak{X})$ is said to have property WCD (CD) if for each $\epsilon > 0$ there exists a weakly (norm) compact convex subset $K_\epsilon \subset \mathfrak{X}$ such that for each $\delta > 0$ there is an index $\tau_0 \in T$ and $E_0 \in B_{\tau_0}$ with $\mu(\Omega - E_0) < \epsilon$ such that for $\tau \geqq \tau_0$

$$\int_E f_\tau d\mu \in \mu(E) K_\epsilon + \delta U$$

$E \subset E_0$, $E \in B_\tau$. Where $U = \{x \in \mathfrak{X} : \|x\| \leqq 1\}$.

The following theorem is believed to be the first theorem which characterizes mean convergent martingales in $L^p(\mu, \mathfrak{X})$.

THEOREM 1. *Let $\{f_\tau, B_\tau, \tau \in T\}$ be a martingale in $L^p(\mu, \mathfrak{X})$ $(1 \leqq p < \infty)$. The net $\{f_\tau, \tau \in T\}$ is convergent in the L^p norm if and only if*

(i) $\sup_{\tau \in T} \|f_\tau\|_p \leqq M < \infty$ *for some M;*

(ii) *$\{f_\tau, \tau \in T\}$ is terminally uniformly integrable; i.e. given an $\epsilon > 0$, there is an index $\tau_0 \in T$ and $\delta > 0$ such that $\tau \geqq \tau_0$ and $\mu(E) < \delta$ imply*

$$\int_E \|f_\tau\| d\mu < \epsilon,$$