A COUNTABILITY CONDITION FOR PRIMARY GROUPS PRESENTED BY RELATIONS OF LENGTH TWO[1]

BY PAUL HILL

Communicated by Dana Scott, October 4, 1968

A subgroup A of the p-primary group G is *nice* if $p^\alpha(G/A) = \{p^\alpha G, A\}/A$ for all ordinals α. We consider the following countability condition: there exists a collection $\mathcal{C}$ of nice subgroups of G such that

(0) $0 \in \mathcal{C}$.

(1) $\mathcal{C}$ is closed with respect to group-theoretic union in G.

(2) If $A \in \mathcal{C}$ and if H is a subgroup of G such that $\{A, H\}/A$ is countable, there exists $B \in \mathcal{C}$ such that $B \supseteq \{A, H\}$ and such that B/A is countable.

The author [1] has referred to this condition as the *third axiom of countability* and has demonstrated that this is the countability condition—not the *first axiom* (countability) nor the *second axiom* (decomposition into a direct sum of countable groups)—which is truly relevant for the proof of Ulm's theorem.

In this note, we outline a short proof of

THEOREM. *Suppose that G is a p-primary group presented by an arbitrary number of generators x_i $(i \in I)$ and relations $R_m (m \in M)$. If each relation R_m involves at most two generators, then G satisfies the third axiom of countability.*

PROOF. There is, of course, no loss of generality in assuming that the index sets I and M both contain an element denoted by 0 and that the relation R_0 is: $x_0 = 0$. By adding repeatedly, if necessary, new generators y_i subject to defining relations of the form $px_i = y_i$, we may assume that for each $i \neq 0$ in I that there exists a relation R_m of the form $px_i = x_j$. Since each element of G has order equal to a power of p, we may in fact assume that given any generator $x_1 \neq x_0$ having order p^n in G that there exists a finite chain $x_1, x_2, x_3, \cdots, x_{n+1} = x_0$ of generators such that $px_i = x_{i+1}$ is one of the given relations. Furthermore, by deleting certain redundancies in both generators and relations we may assume, in the end, that each relation R_m, $m \neq 0$, is precisely of the form $px_i = x_j$ where $i \neq j$. For a quick verification of this, note that if

$$(1) \qquad rx_i = sx_j \quad \text{where } i \neq j \quad \text{and} \quad (r, p) = 1$$

[1] The author acknowledges NSF support under Grant GP-8833.