DIFFERENTIABILITY THEOREMS FOR WEAK SOLUTIONS OF NONLINEAR ELLIPTIC DIFFERENTIAL EQUATIONS[1]

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I shall begin by speaking about the extremals of an integral of the form

$$I(z, G) = \int_G f[x, z(x), \nabla z(x)]dx \tag{1}$$

where G is a domain in R^ν,

$$x = (x^1, \cdots, x^\nu), \qquad z = (z^1, \cdots, z^N), \qquad dx = dx^1 \cdots dx^\nu, \tag{2}$$

$z(x)$ is a vector function, ∇z denotes its gradient which is the set of functions $\{z^i_{,\alpha}\}$ where $z^i_{,\alpha}$ denotes $\partial z^i/\partial x^\alpha$, and $f(x, z, p)$ $(p = \{p^i_\alpha\})$ is generally assumed continuous in all its arguments. The integrals $\int_a^b (1+(dz/dx)^2)^{1/2} dx$ and $\int\int_G [(\partial z/\partial x^1)^2 + (\partial z/\partial x^2)^2] dx^1 dx^2$ are familiar examples of integrals of the form (1) in which $N=1$ in both cases, $\nu = 1$ in the first case, and $\nu = 2$ in the second case and the corresponding functions f are defined, respectively, by

$$f(x, z, p) = (1 + p^2)^{1/2}, \qquad f(x, z, p) = (p_1)^2 + (p_2)^2$$

where we have omitted the superscripts on z and p since $N=1$. The second integral is a special case of the *Dirichlet integral* which is defined in general by

$$D(z, G) = \int_G |\nabla z|^2 dx, \qquad f(x, z, p) = |p|^2 = \sum_{i,\alpha} (p^i_\alpha)^2. \tag{3}$$

Another example is the area integral

$$\begin{aligned} A(z, G) = \int\int_G \Bigg(\left[\frac{\partial(z^2, z^3)}{\partial(x^1, x^2)}\right]^2 &+ \left[\frac{\partial(z^3, z^1)}{\partial(x^1, x^2)}\right]^2 \\ &+ \left[\frac{\partial(z^1, z^2)}{\partial(x^1, x^2)}\right]^2 \Bigg)^{1/2} dx^1 dx^2 \end{aligned} \tag{4}$$

which gives the area of the surface

$$z^i = z^i(x^1, x^2), \qquad (x^1, x^2) \in G, \quad i = 1, 2, 3. \tag{5}$$

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