VECTOR FIELDS ON MANIFOLDS[1]

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INTRODUCTION. We discuss in this paper various topics involving continuous vector fields on smooth differentiable manifolds. In each case the underlying idea is the same: we aim to study geometric properties of manifolds by means of algebraic invariants. The prototype for this is the theorem of H. Hopf [27] on vector fields.

THEOREM OF HOPF. *A compact manifold M has a vector field without zeros if and only if the Euler characteristic of M vanishes.*

Recall that the Euler characteristic of M, χM, is defined by

$$\chi M = \sum_{i=0}^{n} (-1)^i b_i,$$

where $n = \dim M$ and $b_i = i$th Betti number of M ($= \dim$ of $H_i(M; Q)$). Thus the geometric property of M having a nonzero vector field is expressed in terms of the algebraic invariant χM. We will discuss extensions of this idea to vector k-fields, fields of k-planes, and foliations of manifolds.

All manifolds considered will be connected, smooth and without boundary; all maps will be continuous. For background information on manifolds and vector fields see [30], [34] and [67].

1. The index of a tangent k-field. By a *tangent k-field* on a manifold M, we will mean k tangent vector fields $X_1, \cdots, X_k$, which are linearly independent at each point of M. If a k-field is defined at all but a finite number of points, we will say that it is a k-field with *finite singularities*. In this section we discuss an algebraic invariant, the index, which measures whether or not one can alter a k-field so as to remove its singularities.

To define the index we assume that the manifold M has been given a simplicial triangulation so that each point of singularity of the k-field lies in the interior of an m-simplex, where $m = \dim M$. Let p

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