CONSTRAINED EXTREMAL PROBLEMS FOR CLASSES OF MEROMORPHIC FUNCTIONS

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Communicated by Dana S. Scott, October 25, 1968

1. **Introduction.** Let $\mathcal{S}(g)$ denote the class of functions $S(z)$ analytic in $D=\{z: |z|<1\}$ which have the integral representation

$$S(z) = \int_0^{2\pi} g(z, t)dm(t) \tag{1}$$

where $m(t)$ is a nondecreasing function on $0 \leqq t \leqq 2\pi$, $\int_0^{2\pi} dm(t)=1$ and $g(z, t)$ is given for the class. We shall assume that $g(z, t)$ and $g_t(z, t)$ are analytic functions of z in D and Lipschitz continuous with respect to t (uniformly for z in a compact subset of D). G. M. Goluzin has developed a variational method for $\mathcal{S}(g)$ [5] which has proved to be useful in the study of extremal problems within various classes of analytic functions [4], [5], [9], [10].

In this note we present a generalization of the Goluzin variational method. The generalization enables one to preserve side conditions imposed on the functions $m(t)$ in (1). Our work was motivated by the fact that various classes of meromorphic functions have structural formulas based on (1) where $m(t)$ must satisfy the additional condition $\int_0^{2\pi} e^{-it}dm(t)=0$.

Complete proofs and applications of our results will be published elsewhere [8].

2. **Constrained variations.** Let A_ν $(\nu=1, \cdots, n)$ be real numbers and let $u_\nu(t)$ $(\nu=1, \cdots, n)$ be real valued C^1 functions on $0 \leqq t \leqq 2\pi$. The class $\mathcal{S}(g, A)$ is defined to be the class of functions (1) in $\mathcal{S}(g)$ that satisfy the constraints

$$\int_0^{2\pi} u_\nu(t)dm(t) = A_\nu, \qquad \nu = 1, \cdots, n. \tag{2}$$

Let $J(T_1, \cdots, T_n)$ denote the determinant of the matrix $[u_k'(T_j)]$ $(j, k=1, \cdots, n)$ and let $J_\nu = J(T_1, \cdots, T_{\nu-1}, T, T_{\nu+1}, \cdots, T_n)$ $(\nu=1, \cdots, n)$ where $T, T_1, \cdots, T_n \in [0, 2\pi]$. A set of points $T, T_1, \cdots, T_n$ in $[0, 2\pi]$ is said to be *admissible* for $m(t)$ if $m(t)$ is not constant in any neighborhood of each of these points and if the determinants $J(T_1, \cdots, T_n)$, J_ν $(\nu=1, \cdots, n)$ are all nonzero.