CROSS SECTIONALLY SIMPLE SPHERES

BY W. T. EATON

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J. W. Alexander [1] suggested that a 2-sphere S in E^3 is tame if each horizontal cross section is either a point or a simple closed curve. It is not clear whether he presumed that his proof was valid for nonpolyhedral spheres, but his proof implies that there is a homeomorphism h of E^3 onto itself which is invariant on horizontal planes and which takes S onto a round 2-sphere. Bing [6] has described a nonpolyhedral 2-sphere S for which there is no such homeomorphism h.

In this paper we give a proof of Alexander's conjecture. The proof, however, is not elementary as it depends indirectly on Dehn's Lemma [8], Bing's Side Approximation Theorem [2], and Bing's Characterization of tame spheres with homeomorphic approximations in their complementary domains [4].

We assume that S lies exactly between the planes $z=1$ and $z=-1$ and we let $J_t = S \cap \{(x, y, z) \mid z=t\}$ be the horizontal cross section of S at the $z=t$ plane. Note that J_t is a simple closed curve for $-1<t<1$ and J_{-1}, J_1 are points. We let D_t be the disk J_t bounds in the $z=t$ plane. The ϵ-neighborhood of a set X is denoted by $N(X, \epsilon)$, Diam A is the diameter of A, and S^1 stands for the standard 1-sphere. If $-1<\alpha<\beta<1$ and h is a homeomorphism of $S^1 \times [\alpha, \beta]$ into Int S such that

(1) $h(y \times [\alpha, \beta])$ is a vertical line segment for $y \in S^1$, and

(2) $h(S^1 \times t)$ lies in the plane $z=t$ for $t \in [\alpha, \beta]$,

then $A(h, t)$ denotes the annulus in the $z=t$ plane bounded by $h(S^1 \times t)$ and J_t, $S(\alpha, \beta)$ denotes the annulus $S \cap \{(x, y, z) \mid \alpha \leqq z \leqq \beta\}$ and $T(h)$ denotes the torus $h(S^1 \times [\alpha, \beta]) \cup A(h, \alpha) \cup A(h, \beta) \cup S(\alpha, \beta)$.

LEMMA 1. *If $t \in (-1, 1)$ and $\epsilon > 0$ then there are rational numbers α and β and a homeomorphism h: $S^1 \times [\alpha, \beta] \rightarrow$ Int S such that*

(1) $-1<\alpha<t<\beta<1$,

(2) *$h(y \times [\alpha, \beta])$ is a vertical line segment for each $y \in S^1$,*

(3) *$h(S^1 \times r)$ lies in the horizontal plane $z=r$ for $r \in [\alpha, \beta]$,*

(4) *$T(h)$ lies in an ϵ-neighborhood of J_t, and*

(5) *$h(S^1 \times t)$ is homeomorphically within ϵ of J_t.*

PROOF. There is a simple closed curve J in the $z=t$ plane such that $J \subset$ Int S, J is homeomorphically within ϵ of J_t, and the annulus A bounded by J and J_t in the $z=t$ plane lies in $N(J_t, \epsilon)$. J may be moved