COMMUTANTS CONTAINING A COMPACT OPERATOR

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All Hilbert spaces considered in this paper will be complex, and all operators will be bounded and linear. The algebra of all (bounded linear) operators on a Hilbert space $\mathcal{H}$ will be denoted by $\mathcal{L}(\mathcal{H})$. If T is in $\mathcal{L}(\mathcal{H})$ and $\mathfrak{M}$ is a subspace of $\mathcal{H}$ invariant under T then the operator in $\mathcal{L}(\mathfrak{M})$ obtained by restricting T to $\mathfrak{M}$ will be denoted by $T|\mathfrak{M}$.

1. **Introduction.** Let T in $\mathcal{L}(\mathcal{H})$ be a contraction (i.e., $\|T\| \leqq 1$) and suppose that the minimal strong unitary dilation U of T is the bilateral shift of multiplicity one. In [5], Sarason proves two remarkable theorems for such operators. The first (Theorem 1) describes the commutant of T in terms of that of U, while the second (Theorem 2) gives a necessary and sufficient condition that an operator in the commutant of T be compact. In [4], Sz.-Nagy and Foiaş present a generalization of Sarason's first theorem to arbitrary contractions (cf. [2] for an elementary proof of the Sz.-Nagy-Foiaş theorem). Using the Sz.-Nagy-Foiaş result we are able to obtain a generalization of Sarason's second theorem. In this note we state this generalization and briefly outline the proof; complete details will appear elsewhere.

2. **Statement of the theorem.** Throughout, $\mathcal{E}$ will denote a fixed separable Hilbert space and $H^2_{\mathcal{E}}$ will denote the Hilbert space of all weakly measurable, norm square integrable $\mathcal{E}$-valued functions f on the unit circle with the property that the negative Fourier coefficients of f vanish. We will denote by $L^\infty_{\mathcal{L}(\mathcal{E})}$ the space of all essentially bounded, weakly measurable $\mathcal{L}(\mathcal{E})$-valued functions on the unit circle. The space $H^\infty_{\mathcal{L}(\mathcal{E})}$ will be the subspace of $L^\infty_{\mathcal{L}(\mathcal{E})}$ consisting of all functions whose negative Fourier coefficients vanish. For A in $L^\infty_{\mathcal{L}(\mathcal{E})}$, $A^*(e^{it})$ is defined to be $(A(e^{it}))^*$.

Let Θ in $H^\infty_{\mathcal{L}(\mathcal{E})}$ be inner, i.e., let $\Theta(e^{it})$ be unitary for almost all t, and let $\mathcal{H} = H^2_{\mathcal{E}} \ominus \Theta H^2_{\mathcal{E}}$. We consider operators T which may be written in the form

$$T = PU_+|\mathcal{H} \tag{1}$$

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