

# MEASURE THEORETIC GEOMETRY AND ELLIPTIC VARIATIONAL PROBLEMS<sup>1</sup>

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1. This article is intended as a survey of some of the phenomena and some of the recent results associated with higher dimensional boundary value problems in "parametric form" in the calculus of variations. These boundary value problems arise in the following way: Suppose  $m$  and  $n$  are positive integers and one is given a reasonably nice function  $F: R^{m+n} \times \Gamma_m \rightarrow R^+$  where  $\Gamma_m$  denotes the Grassmann manifold of all unoriented  $m$  plane directions in  $R^{m+n}$  (which can be regarded as the space of all unoriented  $m$  dimensional planes through the origin in  $R^{m+n}$ ). If  $S$  is a reasonably nice surface of dimension  $m$  in  $R^{m+n}$ , one defines the integral  $F(S)$  of  $F$  over  $S$  by setting

$$F(S) = \int_{x \in S} F(x, S(x)) dH_m x$$

where  $S(x)$  denotes the tangent  $m$  plane direction to  $S$  at  $x$  and  $H_m$  denotes  $m$  dimensional Hausdorff measure on  $R^{m+n}$ . Hausdorff  $m$  dimensional measure gives a precise meaning to the notion of  $m$  dimensional area in  $R^{m+n}$  and is the basic measure used in defining a theory of integration over  $m$  dimensional surfaces in  $R^{m+n}$  which may have singularities. The Hausdorff  $m$  dimensional measure of a smooth  $m$  dimensional submanifold of  $R^{m+n}$  agrees with any other reasonable definition of the  $m$  area of such a manifold. With this terminology the problem can be stated:

**PROBLEM.** *Among all  $m$  dimensional surfaces  $S$  in  $R^{m+n}$  having a prescribed boundary, is there one minimizing  $F(S)$ ? And if there is, how nice is it?*

To make this problem precise there are, of course, several questions to be answered:

- (1) What is a surface?
- (2) What is the boundary of a surface?
- (3) What are reasonable conditions to put on  $F$ ?

To see what is involved in answering these three questions, it is useful to consider some of the phenomena which arise. For these examples, we fix  $F$  to be identically 1.  $F(S)$  is thus the  $m$  dimensional

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