ASYMPTOTIC PROPERTIES OF ENTIRE FUNCTIONS EXTREMAL FOR THE cos $\pi\rho$ THEOREM

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Let $f(z)$ be an entire function of order $\rho<1$. The classical "cos $\pi\rho$ theorem" of Valiron and Wiman [4, pp. 40, 51] asserts that if

$$\mu(r) = \min_{|z|=r} |f(z)|, \qquad M(r) = \max_{|z|=r} |f(z)|,$$

then, given $\epsilon>0$, the inequality

$$\log \mu(r) > (\cos \pi\rho - \epsilon)\log M(r) \tag{1}$$

holds for a sequence $r=r_n\to+\infty$.

We consider those functions $f(z)$ for which (1) is the best possible inequality, and discuss the global asymptotic behavior of such functions.

THEOREM 1. *Let $f(z)$ be an entire function of order ρ $(0\leqq\rho<1)$, and suppose*

$$\log \mu(r) \leqq [\cos \pi\rho + \epsilon(r)] \log M(r) \tag{2}$$

where $\epsilon(r)\to 0$ as $r\to\infty$.

Then there exists a set E of logarithmic density zero and a slowly varying function[2] *$\psi(r)$ such that*

$$\log M(r) = r^{\rho}\psi(r) \qquad (r \notin E), \tag{3}$$

$$n(r, 0) = [\sin \pi\rho/\pi + o(1)]r^{\rho}\psi(r) \qquad (r\to\infty,\ r\notin E) \tag{4}$$

(where, as usual, $n(r, 0)$ denotes the number of zeros of $f(z)$ in $|z|\leqq r$),

$$\log \mu(r) = [\cos \pi\rho + o(1)]r^{\rho}\psi(r) \qquad (r\to\infty,\ r\notin E\cup H), \tag{5}$$

where H has (linear) density zero.

Further, there exists a real-valued function $\theta(r)$ such that if $k>1$ and $\delta>0$ are given and $\nu(r)$ denotes the number of zeros of $f(z)$ in the region

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[2] A function $\psi(r)$ is said to vary slowly if it is defined and positive for all $r>r_0$ and satisfies $\lim_{r\to\infty} \psi(\sigma r)/\psi(r)\to 1$ $(0<\sigma<\infty)$. For a useful discussion of the properties of such functions see, for example, [9, p. 419]. For a discussion of linear and logarithmic densities see [4, p. 5].