ON GENERALIZED COMPLETE METRIC SPACES

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1. **Introduction.** In reference [3], Luxemburg defined the notion of generalized complete metric space as follows.

DEFINITION. The pair (X, d) is called a *generalized complete metric space* if X is a nonvoid set and d is a function from $X \times X$ to the extended reals satisfying the following conditions:

(D0) $d(x, y) \geqq 0$,

(D1) $d(x, y)=0$ if and only if $x=y$,

(D2) $d(x, y)=d(y, x)$,

(D3) $d(x, y) \leqq d(x, z)+d(z, y)$,

(D4) every d-Cauchy sequence in X is d-convergent, i.e., if $\{x_n\}$ is a sequence in X such that $\lim_{m,n\to\infty} d(x_n, x_m)=0$, then there is an $x \in X$ with $\lim_{n\to\infty} d(x_n, x)=0$.

Some fixed point theorems of the alternative for contractions on such spaces had been proved which include, as a special case, the fixed point theorem of Banach for contractions on complete metric spaces (see [1]). For further information and references, see references [2] and [4].

For convenience, we shall call a pair (X, d) a *generalized metric space* if all but condition D4 of the above definition are satisfied.

Let $\{(X_\alpha, d_\alpha) \mid \alpha \in \mathcal{A}\}$ be a family of disjoint metric spaces. Then there is a natural way of obtaining a generalized metric space (X, d) from $\{(X_\alpha, d_\alpha) \mid \alpha \in \mathcal{A}\}$ in the following manner. Let X be the union of $\{X_\alpha \mid \alpha \in \mathcal{A}\}$. For any $x, y \in X$, define

$$\begin{aligned} d(x, y) &= d_\alpha(x, y) && \text{if } x, y \in X_\alpha && \text{for some } \alpha \in \mathcal{A}, \\ &= +\infty && \text{if } x \in X_\alpha,\ y \in X_\beta && \text{for some } \alpha, \beta \in \mathcal{A} \text{ with } \alpha \neq \beta. \end{aligned}$$

Clearly (X, d) is a generalized metric space. Moreover, if each (X_α, d_α) is also complete, then (X, d) is a generalized complete metric space. The main purpose of this paper is to show that the above procedure is the only way to obtain generalized (complete) metric spaces (see §2). Consequently, most of the fixed points theorems of the alternative on such spaces can be obtained from the corresponding fixed point theorems on metric spaces (see §3).

2. **Decomposition.** Let (X, d) be a generalized metric space. Define a relation $\sim$ on X as follows.

$x \sim y$ if and only if $d(x, y) < +\infty$. Then $\sim$ is obviously an equiva-