HOMOTOPY-EVERYTHING H-SPACES

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Communicated by F. P. Peterson, May 24, 1968

An *H-space* is a topological space X with basepoint e and a *multiplication* map $m: X^2 = X \times X \to X$ such that e is a homotopy identity element. (We take all maps and homotopies in the based sense. *We use k-topologies throughout* in order to avoid spurious topological difficulties. This gives function spaces a canonical topology.) We call X a *monoid* if m is associative and e is a strict identity.

In the literature there are many kinds of H-space: homotopy-associative, homotopy-commutative, A_∞-spaces [3], etc. In the last case part of the structure consists of higher *coherence* homotopies. In this note we introduce the concept of *homotopy-everything H-space* (*E-space* for short), in which all possible coherence conditions hold. We can also define E-maps (see §4). Our two main theorems are Theorem A, which classifies E-spaces, and Theorem C, which provides familiar examples such as BPL. Many of the results are folk theorems. Full details will appear elsewhere.

A space X is called an *infinite loop space* if there is a sequence of spaces X_n and homotopy equivalences $X_n \simeq \Omega X_{n+1}$ for $n \geqq 0$, such that $X = X_0$.

THEOREM A. *A CW-complex X admits an E-space structure with* $\pi_0(X)$ *a group if and only if it is an infinite loop space. Every E-space X has a "classifying space" BX, which is again an E-space.*

1. The machine. This constructs numerous E-spaces.

Consider the category $\mathcal{I}$ of real inner-product spaces of countable (algebraic) dimension and linear isometric maps between them. As examples we have $\boldsymbol{R}^\infty$ with orthonormal base $\{e_1, e_2, e_3, \cdots\}$, and its subspace $\boldsymbol{R}^n$ with base $\{e_1, e_2, \cdots, e_n\}$, which is all there are up to isomorphism. We topologize $\mathcal{I}(A, B)$, the set of all isometric linear maps from A to B, by first giving A and B the *finite* topology, which makes each the topological direct limit of its finite-dimensional subspaces.

LEMMA. *The space* $\mathcal{I}(A, \boldsymbol{R}^\infty)$ *is contractible.*

This is a consequence of two easily constructed homotopies:
(a) $i_1 \simeq i_2: A \to A \oplus A$,
(b) $i_1 \simeq u: \boldsymbol{R}^\infty \to \boldsymbol{R}^\infty \oplus \boldsymbol{R}^\infty$, for some *isomorphism* u.