DIFFERENTIABLY SIMPLE ALGEBRAS

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In this note we announce a result which gives a complete determination of the differentiably simple algebras (in terms of the simple algebras), together with some related results and applications. At the present stage of the theory the algebras considered are for the most part assumed to be finite-dimensional, but otherwise are completely arbitrary (unless expressly stated), i.e. not necessarily associative and not necessarily having a unit element. The main result is new even in the associative case, solves a conjecture of very long standing in the Lie case, and also leads to the solution of one of the principal problems in the theory of power-associative algebras.

Let A be an algebra over a field F. If D is a set of derivations of A (linear transformations d of A into A such that $d(ab)=(da)b+a(db)$ for all a, b in A) then by a *D-ideal* of A is meant an ideal of A invariant under D; A is called *D-simple* if $A^2 \neq 0$ and if A has no proper D-ideals. Also A is called *differentiably simple* if it is D-simple for some D, and hence for the set of *all* derivations of A.

The concept of D-simplicity is particularly important at characteristic p, where there are D-simple algebras which are not simple. Jacobson (see [7]) noted that if F has characteristic p, if S is a simple algebra over F and if $G \neq 1$ is a finite elementary abelian p-group (so G is the direct product of n copies of the cyclic group of order p) then the group ring SG (of G with coefficients in S) is differentiably simple but not simple (and is associative or Lie etc. according as S is); $SG \cong S \otimes_F B_n(F)$, where $B_n(F)$ denotes the (commutative associative) truncated polynomial algebra $F[X_1, \cdots, X_n]/(X_1^p, \cdots, X_n^p)$. ($B_n(F)$ plays an important role, related to the present result, in the theories of simple Lie algebras and certain other algebras at characteristic p.)

THEOREM 1 (MAIN THEOREM). *If A is a finite-dimensional differentiably simple algebra over a field F then either A is simple or the characteristic is p and there are a simple algebra S over F and an elementary abelian p-group $G \neq 1$ such that $A = SG$.*

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