THE COHOMOLOGY OF COMPACT ABELIAN GROUPS

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There are at least two (if not more) cohomology theories for a compact group G. The first is the space cohomology of the topological space underlying G based on Čech cochains, say; the second is an extension of the algebraic cohomology of finite groups based on, say, the Čech cochains with compact support on the classifying space of G. This note is concerned with the second cohomology for compact abelian groups. However, for the sake of completeness we recall the main results of the first theory [2].

THEOREM 1. *If G is a compact abelian group, R a commutative ring with identity, then the Čech cohomology $\check{H}(G, R)$ is a graded commutative Hopf algebra over R and is naturally isomorphic to the Hopf algebra $R\otimes C(G, \mathbf{Z})\otimes\bigwedge(G_0)^{\wedge}$, where $\bigwedge X$ is the integral exterior algebra over the group X in degree 1 with its natural Hopf algebra structure, and where $C(G, \mathbf{Z})$ is the Hopf algebra of all continuous functions from G into the discrete ring $\mathbf{Z}$.*

COROLLARY 2. *If G is a compact connected abelian group and R a commutative ring with identity, then there is a natural isomorphism of commutative Hopf algebras $\check{H}(G, R)\cong R\otimes\bigwedge\hat{G}$.*

DEFINITION 3. Suppose that $E^1(G)\to\cdots E^n(G)\to\cdots E(G)=E^\infty(G)$ is a sequence of spaces and injective maps such that (i) $E^n(G)$ is compact for $n<\infty$, (ii) $\check{H}^i(E^n(G), \mathbf{Z})=0$ for $0<i<n$; $n\leqq\infty$, and that (iii) G acts freely on all spaces. Then $B^n(G)$ is called a classifying space up to n (resp. just a classifying space for $n=\infty$), and for an arbitrary R-module A the graded R-module proj $\lim_n \check{H}(B^n(G), A)$ is independent of the particular choice of a system of classifying spaces. If $A=R$ then the ring structure on $\check{H}(B^n(G), R)$ gives the limit a graded R-algebra structure in a natural fashion. The limit will be called $h(G, A)$.

Functorial constructions for classifying spaces have been given by Milnor, Dold and Lashof, Rothenberg and Steenrod. There is a natural morphism $\check{H}(B(G), A)\to h(G, A)$, but it is not entirely clear whether it is an isomorphism for all compact groups.

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