NOTE ON PRINCIPAL S^3-BUNDLES

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In this note we construct two principal S^3-bundles whose total spaces E_α, E_β are closed smooth manifolds having the properties

(i) E_α, E_β are of different homotopy types, $E_\alpha \not\simeq E_\beta$;

(ii) $E_\alpha \times S^3$, $E_\beta \times S^3$ are diffeomorphic.

The method of construction is a modified dual of that employed in [1] to demonstrate the failure of wedge-cancellation.

Let $a, b \in \pi_n(S^3)$, let B be the classifying space for S^3, and let $\alpha, \beta \in \pi_{n+1}(B)$ be the elements corresponding to a, b respectively. Let $\pi_\alpha: E_\alpha \to S^{n+1}$, $\pi_\beta: E_\beta \to S^{n+1}$ be the bundle projections induced by[1] α, β.

THEOREM 1. *$E_\alpha \simeq E_\beta$ if and only if $\beta = \pm\alpha$ (equivalently, $b = \pm a$).*

PROOF. Sufficiency is obvious, so we suppose $E_\alpha \simeq E_\beta$ and seek to prove $\beta = \pm\alpha$. If $n \leqq 2$, the assertion is trivial. Now there are cell-decompositions

$$E_\alpha = S^3 \cup_a e^{n+1} \cup e^{n+4}, \qquad E_\beta = S^3 \cup_b e^{n+1} \cup e^{n+4}.$$

Thus if $n=3$, a and b are integers and $H_3(E_\alpha) = Z_{|a|}$, $H_3(E_\beta) = Z_{|b|}$, whence $|a| = |b|$. We assume now that $n \geqq 4$ and let $h: E_\alpha \simeq E_\beta$. We may suppose $h(S^3) \subseteq S^3$ and then $h|S^3$ is of degree ± 1. From the exact homotopy sequence we infer that h induces an isomorphism $\pi_{n+1}(E_\alpha, S^3) \cong \pi_{n+1}(E_\beta, S^3)$; these groups are cyclic infinite, generated by i_α, i_β say, so that $h_*(i_\alpha) = \pm i_\beta$. We have a commutative square

$$\begin{array}{ccc} \pi_{n+1}(E_\alpha, S^3) & \overset{h_*}{\cong} & \pi_{n+1}(E_\beta, S^3) \\ \downarrow \partial & & \downarrow \partial \\ \pi_n(S^3) & \cong & \pi_n(S^3) \end{array}$$

where the bottom isomorphism is multiplication by ± 1, $\partial(i_\alpha) = a$, $\partial(i_\beta) = b$. Thus $\pm b = \pm a$ or $\beta = \pm\alpha$.

Let $E_{\alpha\beta} \to E_\alpha$ be induced from π_β by $\pi_\alpha: E_\alpha \to S^{n+1}$, and let $E_{\beta\alpha} \to E_\beta$ be defined similarly.

THEOREM 2. *$E_{\alpha\beta} = E_{\beta\alpha}$. Moreover, $E_{\alpha\beta}$ is equivalent to $E_\alpha \times S^3$ if $\beta \circ \pi_\alpha = 0$ and $E_{\beta\alpha}$ is equivalent to $E_\beta \times S^3$ if $\alpha \circ \pi_\beta = 0$.*

[1] Here and later we deliberately confuse maps and homotopy classes.