SERRE SEQUENCES AND CHERN CLASSES

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Let A be a commutative noetherian ring. A theorem of J.-P. Serre asserts that if P is a finitely generated projective A-module whose rank exceeds the Krull dimension of the maximal ideal space of A, then P has A as a direct summand. We denote, for any prescheme X, the set of closed points of X with the induced topology by X_0. Now given $s_1, \cdots, s_n \in P$, let $F(s_1, \cdots, s_n) = \{x \in \text{Spec}\,(A)_0 : s_1(x), \cdots, s_n(x) \in P/\mathfrak{m}_x P$ are linearly dependent over $A/\mathfrak{m}_x\}$. $F(s_1, \cdots, s_n)$ is a closed subset of Spec $(A)_0$. Implicit in Serre's proof is the stronger result:

THEOREM (SERRE [3]). *Let A be a commutative noetherian ring and let P be a finitely generated projective A-module of rank r. Then there exist $s_1, \cdots, s_r \in P$ such that the codimension of $F(s_p, \cdots, s_r)$ in* Spec $(A)_0$ is $\geqq p$, $1 \leqq p \leqq r$.

This result suggests the following questions.

(A) Let X be a noetherian prescheme and let $\mathcal{E}$ be a locally free coherent $\mathcal{O}_X$-module of rank r. When do there exist global sections $s_1, \cdots, s_r \in \Gamma(X, \mathcal{E})$ such that $\operatorname{codim}_{X_0} F(s_p, \cdots, s_r) \geqq p$, for all p?

A sequence of global sections with these properties will be called a Serre sequence for $\mathcal{E}$. Now, given any sequence $s_1, \cdots, s_r$, of global sections of $\mathcal{E}$, one has the $\mathcal{O}_X$-linear mapping, $(s_p, \cdots, s_r): \mathcal{O}_X^{r-p+1} \to \mathcal{E}$ given locally by $(f_p, \cdots, f_r) \to \sum f_i s_i$. Let $Z_p(s_1, \cdots, s_r)$ be the closed subscheme of X whose structure sheaf is $\operatorname{coker}(\bigwedge^{r-p+1}(s_1, \cdots, s_r)^*)$. We then have a flag, $Z_1(s_1, \cdots, s_r) \supset Z_2(s_1, \cdots, s_r) \supset \cdots \supset Z_r(s_1, \cdots, s_r)$, of subschemes of X, with $Z_p(s_1, \cdots, s_r)_0 = F(s_p, \cdots, s_r)$.

(B) Is the rational equivalence class of $Z_p(s_1, \cdots, s_r)$ independent of the choice of the Serre sequence, $s_1, \cdots, s_r$, for $\mathcal{E}$?

(C) If the answer to (B) is affirmative, what is the significance of these invariants?

Now let A be a fixed commutative noetherian ring, and let X be a prescheme of finite type over A. We say that X is quasi-closed over A if $\pi(X_0) \subset \text{Spec}\,(A)_0$, π being the structure morphism. We say that A is residually infinite if $A/\mathfrak{m}$ is an infinite field for each maximal ideal $\mathfrak{m}$ of A.