DECIDABLE SEMIGROUPS

BY R. C. BUCK[1]

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An extension of the Burnside problem is to determine those presentations of semigroups which, by virtue of the nature of their defining relations, permit a decision algorithm and, in particular, are finite. One such result is that of Green and Rees [2]. The present note deals with another special class of semigroups.

Let $S_{n,r}$ be the semigroup generated by two elements a and b, subject only to the relations:

$$aba = b^n, \tag{1}$$

$$bab = a^r. \tag{2}$$

By symmetry, we may assume $n \leqq r$.

In [1], it was shown that the semigroups $S_{1,r}$ are each finite, with $\operatorname{ord}(S_{1,r}) = 5r+3$ for $r = 1, 2, 3, \cdots$. In partial completion of this, we now have the following:

THEOREM. *The semigroup $S_{2,r}$ is finite for $r = 2, 3, 4$ with*

$$\operatorname{ord}(S_{2,2}) = 31,\ \operatorname{ord}(S_{2,3}) = 74,\ \textit{and}\ \operatorname{ord}(S_{2,4}) \leqq 1390.$$

The semigroup $S_{2,5}$ is infinite, as are all the semigroups $S_{n,r}$ with $n \geqq 3$.

This result leaves undetermined the exact order of $S_{2,4}$, the nature of $S_{2,r}$ for $r \geqq 6$, and the question of whether a decision algorithm exists for the infinite semigroups; the case $n = 2$, $r = 5$ is somewhat special, and it may be possible that there are some that are finite with $r \geqq 6$.

We also note that the results in this theorem form an excellent test problem for persons interested in theorem proving by computer.

We outline only the proof of finiteness for the case $n = 2$, $r = 3$. The proof proceeds by deriving a sequence of further relations from (1) and (2), which suffice to reduce any word in a and b to one of the words in the following list:

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