STRONG CARLEMAN OPERATORS ARE OF HILBERT-SCHMIDT TYPE

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This is the solution of a problem posed by G. Targonski [6, §13 V]: Do unbounded strong Carleman operators exist? In fact, we shall prove that every strong Carleman operator is a Hilbert-Schmidt operator (hence it is certainly bounded).

1. Definitions and known results. Carleman operators are usually defined in the space $L_2(a, b)$ where $a \geqq -\infty$, $b \leqq \infty$; without any restriction of generality we may assume that $-\infty < a < b < \infty$. There are several definitions of a Carleman operator used in the literature (e.g. T. Carleman [1], M. Stone [5], G. Targonski [6]; for "semi-Carleman operators" see M. Schreiber [4]). We shall mainly follow the definition used by G. Targonski, but in addition to his definition we shall assume that a Carleman operator is densely defined (e.g. §3).

DEFINITIONS. A densely defined operator K in the Hilbert space $L_2(a, b)$ is called a *Carlemann operator* if it allows a representation of the form

$$(Kf)(x) = \int_a^b K(x, y)f(y)dy \quad \text{for almost all } x,$$

where $\int_a^b |K(x, y)|^2 dy < \infty$ for almost all x. The domain of K consists of all elements $f \in L_2(a, b)$ such that $\int_a^b K(x, y)f(y)dy$ (which is defined for almost all x) represents an element of $L_2(a, b)$.

An operator K is called a *strong Carleman operator* if UKU^* is a Carleman operator for every unitary operator U. An operator K in a Hilbert space is a *Hilbert-Schmidt operator* (or K is of Hilbert-Schmidt type) if for every orthonormal system (ϕ_n), $\sum_n |K\phi_n|^2 < \infty$ (this series has the same value for all complete orthonormal systems).

It is known (e.g. [6]) that every Hilbert-Schmidt operator is a strong Carleman operator. In [6] it is also shown that bounded strong Carleman operators are of Hilbert-Schmidt type. Using the result of this note we may say: *An operator in $L_2(a, b)$ is a strong Carleman operator if and only if it is a Hilbert-Schmidt operator.*

We shall use the following known results:

THEOREM I ([6, LEMMATA 9.1 AND 9.2]). *If K is a strong Carleman operator and B is bounded, then BK and KB are strong Carleman operators.*