THE AXIOM OF DETERMINATENESS AND REDUCTION PRINCIPLES IN THE ANALYTICAL HIERARCHY

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Let R be the set of all sets of natural numbers. A collection $\mathcal{A}$ of subsets of R satisfies a *reduction principle* if, for every A and $B \in \mathcal{A}$, there are A' and $B' \in \mathcal{A}$ such that $A' \subseteq A$, $B' \subseteq B$, $A' \cup B' = A \cup B$, and $A' \cap B'$ is empty. For $n > 0$ let Π^1_n and Σ^1_n be, respectively the set of Π^1_n subsets of R and the set of Σ^1_n subsets of R. It is known that Π^1_1 and Σ^1_2 satisfy reduction principles and that for no n do both Π^1_n and Σ^1_n satisfy reduction principles. (For basic definitions and facts concerning the analytical hierarchy and the degrees of unsolvability, see [5].) Using the Axiom of Constructibility, Addison [1] shows that, for all $n \geqq 2$, Σ^1_n satisfies a reduction principle. J. Silver has shown that Addison's result is consistent with the assertion that a measurable cardinal exists.

For each $n > 0$, let Γ^1_n be Π^1_n if n is odd and Σ^1_n if n is even. For a statement of the Mycielski-Steinhaus Axiom of Determinateness (AD) and proofs of some of its consequences, see [4]. We assume AD and the Axiom of Dependent Choice (DC) and outline a proof that, for every n, Γ^1_n (and hence $\mathbf{\Gamma}^1_n$) satisfies a reduction principle. This result has been proved independently by Moschovakis and Addison [2].

Since AD is false, a word is in order about the significance of our proof. In the notation of [4], AD says that, for every $P \subseteq 2^\omega$, $G_2(P)$ is determined. Although this contradicts the Axiom of Choice, it remains possible that a very large class of $G_2(P)$ are determined. For instance, it is possible that $G_2(P)$ is determined for every projective P, and this is enough to deduce our result. Indeed, to prove reduction for Γ^1_n we need only assume that $G_2(P)$ is determined for every $\mathbf{\Delta}^1_{n-1}P$. We need DC for $n \geqq 4$. While AD may well be consistent with DC, our justification for using DC is rather that we are assuming only a part of AD which we hope to be consistent with the Axiom of Choice.

Our tool in studying the analytical hierarchy is the Lemma below. Our first proof of reduction for Π^1_3 was based on a new proof by Blackwell [3] using infinite games—of reduction for Π^1_1. (The methods of [2] are closely related to those of Blackwell.) However, the Lemma provides a different proof which generalizes easily to all odd levels of the hierarchy. The Lemma is a consequence of AD and is an interesting proposition in its own right. Also, the problem of proving the Lemma consistent (say, assuming large cardinals of some kind) might