ON THE BOUNDARY POINT PRINCIPLE FOR ELLIPTIC EQUATIONS IN THE PLANE[1]

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1. Let D be an open connected subset of E^n $(n \geqq 2)$ and denote by $\mathcal{L}_\alpha(D)$ the class of second order uniformly elliptic operators of the form $L = \sum_{i,j=1}^n a_{ij}\partial^2/\partial x_i \partial x_j$ with coefficients defined in D and satisfying there the condition $\sum_{i,j=1}^n a_{ij}\xi_i\xi_j \geqq \alpha \sum_{i=1}^n \xi_i^2$, for some constant α in the range $0 < \alpha \leqq 1/n$, and the normalization $\sum_{i=1}^n a_{ii} = 1$. It is well known [1]–[4] that such differential operators enjoy the following strong minimum and boundary point principles: A nonconstant, twice differentiable function $u(x)$, satisfying $Lu \leqq 0$ in D, cannot attain a local minimum in D. Moreover if u attains a local minimum at a boundary point x^0 where ∂D has the inner sphere property, and if ν is a unit vector directed internally to the sphere, then

$$\liminf_{t \to 0^+} \left\{ \frac{u(x^0 + t\nu) - u(x^0)}{t} \right\} > 0.$$

Equivalently, the boundary point principle states that for $\|x - x^0\|$ sufficiently small there exists a *positive* constant m (depending upon ν) such that

$$u(x) \geqq u(x^0) + m\|x - x^0\|$$

along the line $x^0 + t\nu$. In this note we wish to obtain, for the case of a plane domain $(n=2)$, an analogous lower bound for the approach of $u(x)$ to a minimum occurring on the boundary when the inner sphere property is replaced by an inner cone (sector) property. The proof is based upon a comparison with a barrier function which has recently been obtained [5] for the class $\mathcal{L}_\alpha$ in a plane sector with the aid of elliptic extremal operators [6]. Our result is the best possible for the class of differential operators $\mathcal{L}_\alpha$ and moreover shows explicitly the dependence upon the ellipticity constant α.

2. We shall first describe our barrier function for the plane sector

$$S(\theta_0) = \{(x, y) : r > 0, \ |\theta| < \theta_0 < \pi\}$$

where r, θ denote the polar coordinates of the point (x, y).

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