NONLINEAR EIGENVALUE PROBLEMS AND GALERKIN APPROXIMATIONS

BY FELIX E. BROWDER

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Let X be a reflexive Banach space, T and S two mappings of X into its conjugate space X^*. We denote the pairing between w in X^* and u in X by (w, u), and weak convergence (in either X or X^*) by $\rightharpoonup$, strong convergence (in either X or X^*) by $\rightarrow$.

By an eigenvalue problem for the pair (T, S), we mean the problem of finding an element u in X and a real number λ such that

$$T(u) = \lambda S(u), \tag{1}$$

with u possibly satisfying additional normalization conditions. It is our purpose in the present note to describe a way of applying a method of Galerkin type to such problems which works in particular for nonlinear elliptic boundary value problems of variational type. We obtain from it a general theorem on the existence of normalized eigenfunctions for the latter problem, and in the case of T and S odd operators, we obtain also an extremely general form of a theory of Lusternik-Schnirelman type guaranteeing the existence of infinitely many distinct normalized eigenfunctions.

We consider first some restrictions that may be placed on the nonlinear operator T.

DEFINITION 1. *T is said to satisfy condition (S) if for any sequence $\{u_j\}$ in X with $u_j \rightharpoonup u$ in X and $(T(u_j)-T(u), u_j-u)\rightarrow 0$, we have $u_j\rightarrow u$ in X.*

DEFINITION 2. *T is said to satisfy condition $(S)_0$ if for each sequence $\{u_j\}$ in X with $u_j\rightharpoonup u$ in X, $T(u_j)\rightharpoonup z$ in X^*, and $(T(u_j), u_j)\rightarrow(z, u)$, we have $u_j\rightarrow u$ in X.*

LEMMA 1. (a) *If T satisfies condition (S), it satisfies condition $(S)_0$.*

(b) *If T is continuous and satisfies condition $(S)_0$, and if K is any compact set of X^*, B any bounded closed set of X, then $T^{-1}(K)\cap B$ is compact.*

(c) *If T is continuous and satisfies condition $(S)_0$, then the image under T of any bounded closed set B of X is closed in X^*.*

PROOF OF LEMMA 1. PROOF OF (a). Suppose $u_j\rightharpoonup u$, $T(u_j)\rightharpoonup z$, and $(T(u_j), u_j)\rightarrow(z, u)$. Then