A FIXED POINT THEOREM FOR SET VALUED MAPPINGS[1]

BY JACK T. MARKIN

Communicated by C. B. Morrey, Jr., February 5, 1968

Let H be a real Hilbert space with closed unit ball B and let $K(H)$ denote the family of nonempty compact convex subsets of H supplied with the Hausdorff metric D generated by the norm of H. A mapping ϕ: $H \to K(H)$ is *contractive* if for any pair x, $y \in H$, $D(\phi(x), \phi(y)) \leqq D(x, y)$. If $x \in \phi(x)$, then x is a fixed point of ϕ.

In this paper we shall prove the following fixed point theorem for set valued contractions, which is an extension of a theorem of Browder [1].

THEOREM 1. *Let* ϕ: $H \to K(H)$ *be a contractive mapping such that* $\phi(x) \subset B$ *for every* $x \in B$. *Then* ϕ *has a fixed point in* B.

The proof relies on a generalization of the concept of monotone mappings of H into H to mappings of H into $K(H)$, and also depends on Theorem 2 which we state without proof.

THEOREM 2. *Assume that* X *is a complete bounded metric space and that* ϕ *maps* X *into the family of nonempty closed subsets of* X. *If there is a* $k \in [0, 1)$ *such that for any pair* x, $y \in X$, $D(\phi(x), \phi(y)) \leqq kD(x, y)$, *then* ϕ *has a fixed point.* (*Here* D *is the Hausdorff metric generated by the metric of* X.)

A mapping G of H into the family of nonempty subsets of H is *monotone* if given u, $v \in H$ and $\bar{u} \in G(u)$ there is a $\bar{v} \in G(v)$ such that $(\bar{u}-\bar{v},\ u-v) \geqq 0$.

LEMMA 1. *Let* G: $H \to K(H)$ *be a continuous monotone map, and assume that for some pair* v, $\bar{v} \in H$ *and every* $u \in H$ *there is a* $\bar{u} \in G(u)$ *such that* $(\bar{v}-\bar{u},\ v-u) \geqq 0$. *Then,* $\bar{v} \in G(v)$.

PROOF. Suppose $\bar{v} \notin G(v)$. By weak compactness, there is a $w \in H$ such that $(\bar{v}, w) < (x, w)$ for every $x \in G(v)$. Let $v_n = v-(1/n)w$; since G is monotone there exists a $\bar{v}_n \in G(v_n)$ such that $(\bar{v}-\bar{v}_n,\ v-v_n) \geqq 0$ for all n. Therefore $(\bar{v}, w) \geqq (\bar{v}_n, w)$. Since $D(G(v_n), G(v))$ tends to 0 as $n \to \infty$ by continuity, we may assume that $\{\bar{v}_n\}$ tends weakly to a point $\bar{x}$ and that there is a sequence $\{z_n\}$, $z_n \in G(v)$, such that

[1] This work was supported by the U. S. Army Research Office-Durham, contract DA-31-124-ARO-D-265.