BOUNDS AND MAXIMAL SOLUTIONS FOR NONLINEAR FUNCTIONAL EQUATIONS

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1. **Introduction.** Consider the two functional equations

$$Lu = Nu + p, \tag{1}$$

$$Lu = Mu + q \tag{2}$$

in a real Banach space B, where L is a linear operator mapping a subset of B into B; M and N are operators (in general, nonlinear) mapping B into B; p and q are fixed elements in B; and the following inequality holds:

$$Nu + p \leqq Mu + q \quad \text{for all } u \in B. \tag{3}$$

Here "$\leqq$" signifies a partial ordering induced by a cone $K \subset B$ [**3**] of "positive" elements:

$$u \leqq v \quad \text{if and only if} \quad (v - u) \in K.$$

In this paper we extend results obtained previously [**2**] for positive solutions (that is, solutions in K) of (1) and (2); here we consider solutions which are not necessarily in K. Specifically, under condition (3) and certain other assumptions, we establish below that the (unique) solution of (2) is an upper bound on all solutions of (1) (§2); and, under additional hypotheses on N, we construct the "maximal" solution for (1) (§3). Finally, we make some remarks about positive solutions (§4).

Applications of these results to systems of nonlinear equations and nonlinear boundary value problems for ordinary differential equations can be found in [**1**], [**2**]. Related results in the case of (elliptic) partial differential equations have been obtained by Parter [**4**].

The result in §2 might be described as a generalization of the "generalized Bellman's Lemma" (see [**5**]); for it follows from (3) that any solution of (1) satisfies $Lu \leqq Mu + q$, and integral inequalities of this form are treated in [**5**].

We make the following assumptions once and for all:

(A_1) L has a bounded inverse L^{-1} which is defined on B and leaves the cone K invariant.

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