A CHARACTERIZATION OF SPACES WITH VANISHING GENERALIZED HIGHER WHITEHEAD PRODUCTS

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A subject of recent investigation in homotopy theory has been the study of generalized higher order Whitehead products, cf. [1] and [3]. Let us say that a space, X, has property P_n if for any $f_1, \cdots, f_n$, $f_i: SA_i \to X$, we have $0 \in [f_1, \cdots, f_n]$, where $[f_1, \cdots, f_n]$ denotes the set of all nth order Whitehead products of $f_1, \cdots, f_n$, as defined in [3]. Thus $0 \in [f_1, \cdots, f_n]$ means that $f_1 \vee \cdots \vee f_n$: $SA_1 \vee \cdots \vee SA_n \to X$ can be extended to some F: $SA_1 \times \cdots \times SA_n \to X$. (We note at this point that it is an unresolved conjecture as to whether X has property P_n implies that 0 is the *only* element of $[f_1, \cdots, f_n]$.) Now if X is an H-space, then X possesses property P_n for all n, [3]. Thus multiplicative properties of X itself are too strong to distinguish among the various properties P_n. On the other hand, it follows from results of [1] and [4] that a space has property P_2 if and only if its loop space is homotopy-commutative. In Theorem 1 below, we shall extend this result to characterize those spaces which have property P_n in terms of higher homotopy-commutativity properties of their loop spaces. Since we shall wish to be able on occasion to replace a loop space by an equivalent CW-monoid, we shall restrict our attention to the category of countable CW-complexes.

The higher homotopy-commutativity properties we need are described in the following definition which was introduced in [7].

DEFINITION. An associative H-space, Y, is called a C_n-*space* provided that there exist maps Q_i: $C(i-1) \times Y^i \to Y$, $1 \leq i \leq n$, such that:

(1) Q_1: $C(0) \times Y \to Y$ is the identity;

(2) $Q_i([\mu, \nu] \circ d_p(r, s), y_1, \cdots, y_i) = Q_p(r, y_{\mu(1)}, \cdots, y_{\mu(p)}) \cdot Q_q(s, y_{\nu(1)}, \cdots, y_{\nu(q)})$, for (p, q)-shuffles (μ, ν), where $p+q=i$, $r \in C(p-1)$, and $s \in C(i-p)$; and

(3) if e denotes the identity of Y, and if $y_i = e$, then

$$Q_i(T, y_1, \cdots, y_i) = Q_{i-1}(D_j(T), y_1, \cdots, \hat{y}_j, \cdots, y_i).$$

Here $C(i)$ is the convex linear cell described in [2], namely the convex hull of the orbit of the point $(1, \cdots, n+1)$ under permutation of the coordinates in R^{n+1}. The map d_p: $C(p-1) \times C(i-p) \to C(i)$ is given by $d_p(x_1, \cdots, x_p, y_1, \cdots, y_{i-p+1}) = (x_1, \cdots, x_p, y_1+p, \cdots, y_{i-p+1}+p)$,

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