THE ANGLE OF AN OPERATOR AND POSITIVE OPERATOR PRODUCTS[1]

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1. Introduction. Let $A \geqq 0$ and $B \geqq 0$ be two positive bounded selfadjoint operators. The two algebraic questions which immediately arise are: (1) is $A+B \geqq 0$; (2) is $BA \geqq 0$? The first question (and its extension to accretive operators on a Banach space) has trivially an affirmative answer; the second question also has an affirmative answer, *under the additional condition that A and B commute.* However, apparently question (2), which due to the general nonselfadjointness of BA must be reformulated as (2′) $\operatorname{Re}(BAx, x) \geqq 0$ for all x, has remained unanswered for general (i.e., *noncommuting*) operators A and B. This is clearly an important mathematical question, and the purpose of this announcement is to state sufficient (and rather sharp) conditions for the more general (2″) $\operatorname{Re}[BAx, x] \geqq 0$ for all x, where $[x, y]$ is a semi-inner product on any Banach space (in the special case of a Hilbert space, it is necessarily the inner product).

In §2 we introduce two new quantities which are technically essential in our treatment, namely the angle of an operator and the minimum of a certain (norm) function related to tangent functionals. In §3 we bound the behavior of the latter, determining exact values for the important class of selfadjoint operators. Using these two quantities, in §4 we give criteria for (2″) to hold; our main result is Theorem 4.2. In §5 we give positive lower bounds for the resulting accretive operator products, and compare them with bounds for commuting selfadjoint operators. In §6 we apply these results to a semigroup question which motivated this work. However, it is expected that the criteria here developed will be useful elsewhere in operator theory and in those parts of theoretical physics where a zero commutator is the exception. In §7 we make further comments. Complete proofs of these and related results may be found in the references (in particular, [6]) and in a paper under preparation.

2. Definitions, the angle of an operator. Consider the following (well-known) quantitative functions defined on the Banach algebra of bounded operators $B: X \to X$, X a complex Banach space, $[x, y]$ a semi-inner product (see Lumer [11]), $\|x\|=1$: $\|B\| = \sup\|Bx\|$;

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