REMARKS ON THE HEAWOOD CONJECTURE (NONORIENTABLE CASE)

BY J. W. T. YOUNGS

Communicated by J. D. Swift, August 22, 1967

1. **Introduction.** In 1890 Heawood [3] stated a problem which has become known as the Heawood map-coloring conjecture. It is concerned with the chromatic number of 2-manifolds, and is one of the oldest problems in combinatorics.

The conjecture for orientable manifolds is not yet completely solved, though considerable progress has been made in the past few years. On the other hand, the companion conjecture for nonorientable manifolds was solved completely by Ringel [4] during the years 1953–1959. For this purpose he developed a theory of "leading" permutations but is hardly pleased with the complications involved. The object of this note is to announce a successful attack on the problem using quite simple combinatorial techniques.

2. **Definitions, theorems and conjectures.** All manifolds $\tilde{M}$ are closed, nonorientable and of dimension 2. Each $\tilde{M}$ has a standard model which is a 2-sphere with added cross caps. $\gamma(\tilde{M})$, the *genus* of $\tilde{M}$, is the number of cross caps. K_n is the complete n-graph, that is, the graph with n vertices where each pair of distinct vertices is joined by exactly one arc. $\tilde{\gamma}(K)$, the nonorientable genus of K, is the smallest integer k such that the graph K can be topologically imbedded in a manifold of genus k. The chromatic number of $\tilde{M}$ is designated by $\mathrm{ch}(\tilde{M})$. Define

$$\tilde{H}(q) = [(7 + (1 + 24q)^{1/2})/2],^1 \qquad q = 1, 2, 3, \cdots,$$
$$\tilde{I}(n) = \{(n-3)(n-4)/6\}, \qquad n = 5, 6, 7, \cdots.$$

The Heawood theorem states that

$$\mathrm{ch}(\tilde{M}) \leqq \tilde{H}(\gamma(\tilde{M})) \tag{1}$$

and the *Heawood conjecture* is that equality holds in (1).

The *complete graph theorem* is that

$$\tilde{\gamma}(K_n) \geqq \tilde{I}(n), \tag{2}$$

and the *complete graph conjecture* is that equality holds in (2).

[1] $[a]$ is the largest integer not greater than a, and $\{a\}$ is the smallest integer not less than a.