BERRY-ESSEEN BOUNDS FOR THE MULTI-DIMENSIONAL CENTRAL LIMIT THEOREM[1]

BY R. N. BHATTACHARYA

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1. Introduction. Let $\{X_n\}$ be a sequence of independent and identically distributed random variables each with mean zero, variance unity, and finite absolute third moment β_3. Let F_n denote the distribution function of $(X_1 + \cdots + X_n)/(n)^{1/2}$. Berry [2] and Esseen [4] have proved that

$$(1)\quad \sup_{x \in R_1} \left| F_n(x) - \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{x} e^{-y^2/2} dy \right| \leqq c\beta_3/n^{1/2}, \qquad n = 1, 2, \cdots,$$

where c is a universal constant. Consider now a sequence $\{X(n) = (X_1^{(n)}, \cdots, X_k^{(n)})\}$ of independent and identically distributed random vectors in R_k each with mean vector $(0, \cdots, 0)$ and covariance matrix I, the $k \times k$ identity matrix. If P_n denotes the probability distribution of $(X^{(1)} + \cdots + X^{(n)})/n^{1/2}$ and Φ is the standard k-dimensional normal distribution, then it is well known that P_n converges weakly to Φ as $n \to \infty$. Bergström [1] has extended (1) to this case, assuming finiteness of absolute third moments of the components of $X^{(1)}$. Since weak convergence of a sequence Q_n of probability measures to Φ means that $Q_n(B) \to \Phi(B)$ for every Borel set B satisfying $\Phi(\partial B) = 0$, ∂B being the boundary of B, it seems natural to seek bounds of $|P_n(B) - \Phi(B)|$ for such sets B (called *Φ-continuity sets*). Let $\mathcal{A}$ be a class of Borel sets such that, whatever be the sequence Q_n converging weakly to Φ, $Q_n(B) \to \Phi(B)$ as $n \to \infty$ uniformly for all $B \in \mathcal{A}$. Such a class is called a *Φ-uniformity* class. By a theorem of Billingsley and Topsoe [3], a class $\mathcal{A}$ is a Φ-uniformity class if and only if $\sup\{\Phi(\partial B)^\epsilon; B \in \mathcal{A}\} \downarrow 0$ as $\epsilon \downarrow 0$, where $(\partial B)^\epsilon$ is the ϵ-neighborhood of ∂B. This leads one naturally to consider the class $\mathcal{A}_1(d, \epsilon_0)$ of all Borel sets B for which $\Phi(\partial B)^\epsilon \leqq d\epsilon$ for $0 < \epsilon < \epsilon_0$, d and ϵ_0 being any two given positive constants. One may also consider the class $\mathcal{A}_1^*(d, \epsilon_0)$, which is the largest translation-invariant subclass of $\mathcal{A}_1(d, \epsilon_0)$; this means that $B \in \mathcal{A}_1^*(d, \epsilon_0)$ if and only if all translates of B belong to $\mathcal{A}_1(d, \epsilon_0)$.

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