THE IMPOSSIBILITY OF FILLING E^n WITH ARCS

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The purpose of this paper is to outline a proof of the following

MAIN THEOREM. *If f is a closed continuous map of E^n onto any space S, then some point in S has an inverse image which is not an arc.*

In 1936 J. H. Roberts [1] showed that there does not exist an upper semicontinuous (usc) collection of arcs filling the plane. Recently L. B. Treybig [2] has obtained some partial results for polygonal arcs in E^n. In 1955 Eldon Dyer [3] outlined a proof that there is no continuous decomposition of E^n into arcs. This proof incorporates some of the ideas of both Roberts and Dyer.

We will suppose that all statements are for E^n for a given n.

DEFINITIONS. If U and V are sets with disjoint closures, we say that an arc α has k *folds* between U and V if α contains $k+1$ disjoint subarcs between U and V. Furthermore, if the distance between each pair of the $k+1$ subarcs is greater than ϵ, we say that the *width* of the folds is greater than ϵ. If α contains a subarc which has endpoints in U and which intersects V, then α is said to have a fold with the *bend* in V.

If K is a set, $\epsilon>0$, let $N_\epsilon(K)$ denote the open ϵ-neighborhood of K in E^n. If H is a collection of sets, let H^* denote the set of all points covered by elements of H.

Suppose A is compact and B is a closed subset of A. If any two points of E^n-A which are separated by A are also separated by B, then B is said to be *essential* in A. If H is a usc collection of arcs and points filling A and B intersects each element of H, then B is said to be *full* in A^H. If B meets each element of H in a continuum, then B is said to be a *quasi-section* of A^H.

Assume H is a usc collection of arcs and points filling the compact set X.

LEMMA 1. *If Y is a quasi-section of X^H then Y is essential in X.*

The proof is an exercise in the Vietoris mapping theorem on the Čech homologies of X, Y, and the decomposition space.

LEMMA 2. *If K is full in X^H, U is open, $\overline{U}\cap K=\varnothing$, and no element*

[1] The results presented in this paper are a part of the author's Ph.D. thesis at the University of Wisconsin, written under the direction of Professor R. H. Bing.